WIT

BSc in Applied Computing
BSc in Computer Forensics
BSc in Entertainment Systems
BSc in the Internet of Things

Examination:

Discrete Mathematics
SEMESTER — 1

December 2017 DURATION: 2 Hours

Internal Examinar: Dr Denis Flynn
Dr Kieran Murphy

External Examinar:

Date:

Time:
Venue:

Instructions to Candidates

1. Answer all Questions.
2. Graph Paper and Statistical Tables Required.
3. Exam Paper (4 pages) and Formula Sheet (1 page)

Question 1

(a) Let $A = \{0, 2, 3\}$, $B = \{2, 3\}$, and $C = \{1, 5, 9\}$. Determine which of the following statements are true. Give reasons for your answers.

(i) $3 \in A$ **(iii)** $\{3\} \subseteq A$ **(v)** $A \subseteq B$

(ii) $\{3\} \in A$ **(iv)** $B \subseteq A$ **(vi)** $\emptyset \subseteq C$

(6 marks)

(b) Construct an expression corresponding to the following argument and hence or otherwise determine whether the argument is valid.

"*If it is raining, it is not cold*"
"*If it is not raining, John is not wearing a coat*"
"*It is cold*"

$\therefore$ "*John is not wearing a coat*"

(6 marks)

(c) Let R be the relation on the set $A = \{1, 2, 3, 4, 5, 6\}$ where $(a, b) \in R$ iff a and b are the same length when written in English.

(i) Represent R using a digraph.

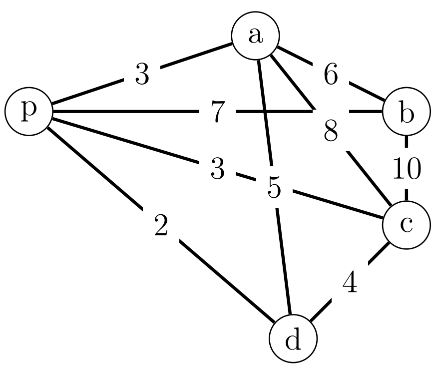
(ii) Is R reflexive? symmetric? transitive?

(iii) Is R an equivalence relation? and if yes, what the resulting equivalence classes?

(7 marks)

(d) A company is considering building a gas pipeline to connect 4 wells (a, b, c and d) to a process plant p. The possible pipelines that they can construct and their costs (in millions of euro) are shown in the accompanying graph.

What pipelines do you suggest be built and what is the total cost of your suggested pipeline network? Show the working of the algorithm used.



(6 marks)

Question 2

(a) In the following Python code three functions are implemented. The purpose of these functions is to check various properties of the relation `relation`, from the set `source` to the set `target`. For each function, identify the property being tested and explain your answer.

```python
def isPropertyA(source, target, relation):
    T = {pair[1] for pair in relation}
    return len(T)==len(relation)


def isPropertyB(source, target, relation):
    T = {pair[1] for pair in relation}
    return T==target


def isPropertyC(source, target, relation):
    S = {pair[0] for pair in relation}
    T = {pair[1] for pair in relation}
    return S==source and len(S)==len(relation) and T.issubset(target)
```

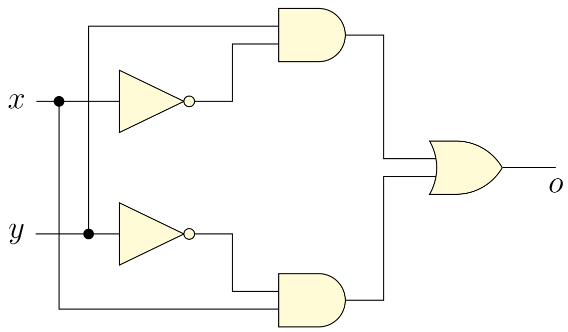
(7 marks)

(b) Consider the following logical circuit with two inputs, and single output.

(i) Construct a logical expression to represent this circuit.

(ii) Is there an input case for which the output is on?

(iii) Hence, construct a logical circuit or equivalent expression, containing three inputs, x, y, and z, for which the output is on when exactly one input is on.



(10 marks)

(c) Prove that every prime number greater than 3 is either one more or one less than a multiple of 6.

Hint. Prove the contrapositive by cases. **(8 marks)**

Question 3

(a) Consider the sequence

$$5, 9, 13, 17, 21, \ldots$$

(i) Construct a recursive definition for the sequence.

(ii) Construct a closed formula for the nth term of the sequence.

(iii) Is 2017 a term in the sequence? Explain.

(4 marks)

(b) A graph, G, has adjacency matrix $A = \begin{pmatrix} 0 & 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$

(i) Is G a simple graph?

(ii) State the degree sequence of G.

(iii) How many edges does G have?

(6 marks)

(c) Let $S = \{1, 2, 3, 4, 5, 6\}$

(i) How many subsets are there of cardinality 4?

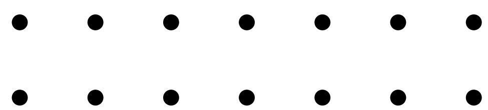
(ii) How many subsets of cardinality 4 have $\{2, 3, 5\}$ as a subset?

(iii) How many subsets of cardinality 4 contain at least one odd number?

(iv) How many subsets of cardinality 4 contain exactly one even number?

(8 marks)

(d) Consider the follow diagram, consisting of two rows of seven dots.



How many

(i) Squares **(ii)** Quadrilaterals

can be drawn using the dots as vertices (corners).

(7 marks)

Question 4

(a) Consider the function $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4\}$ given by the table below:

x	1	2	3	4	5
$f(x)$	3	2	4	1	2

(i) Is f injective? Explain.

(ii) Is f surjective? Explain.

(4 marks)

(b) Suppose that U is an infinite universal set, and A and B are infinite subsets of U. Answer the following questions with a brief explanation.

(i) Must $\overline{A}$ be finite?

(ii) Must $A \cup B$ infinite?

(iii) Must $A \cap B$ be infinite?

(6 marks)

(c) Evaluate the following series:

(i) $\sum_{i=1}^{3}(2 + 3i)$

(ii) $\sum_{k=1}^{n}(2k - 1)$ for $n = 1, 2, 3, 4$

(8 marks)

(d) If G is a simple graph with at least two vertices, prove that G must contain two or more vertices of the same degree. **(7 marks)**

Laws of Logic

Logical Connective	**Symbol**	**Python Operator**	**Precedence**	**Logic Gate**
Negation (NOT)	$\neg$	**not**	Highest	
Conjunctive (AND)	$\wedge$	**and**	Medium	
Disjunctive (OR)	$\vee$	**or**	Lowest	

Basic Rules of Logic

Commutative Laws

$$p \vee q \Leftrightarrow q \vee p \qquad p \wedge q \Leftrightarrow q \wedge p$$

Associative Laws

$$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r) \qquad (p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$$

Distributive Laws

$$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r) \qquad p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$$

Identity Laws

$$p \vee \mathbf{F} \Leftrightarrow p \qquad p \wedge \mathbf{T} \Leftrightarrow p$$

Negation Laws

$$p \wedge (\neg p) \Leftrightarrow \mathbf{F} \qquad p \vee (\neg p) \Leftrightarrow \mathbf{T}$$

Idempotent Laws

$$p \vee p \Leftrightarrow p \qquad p \wedge p \Leftrightarrow p$$

Null Laws

$$p \wedge \mathbf{F} \Leftrightarrow \mathbf{F} \qquad p \vee \mathbf{T} \Leftrightarrow \mathbf{T}$$

Absorption Laws

$$p \wedge (p \vee q) \Leftrightarrow p \qquad p \vee (p \wedge q) \Leftrightarrow p$$

DeMorgan's Laws

$$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q \qquad \neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$$

Involution Law

$$\neg(\neg p) \Leftrightarrow p$$

Implications and Equivalences

Detachment (Modus Ponens)

$$(p \to q) \wedge p \Rightarrow q$$

Indirect Reasoning (Modus Tollens)

$$(p \to q) \wedge \neg q \Rightarrow \neg p$$

Disjunctive Addition

$$p \Rightarrow (p \vee q)$$

Conjunctive Simplification

$$(p \wedge q) \Rightarrow p \qquad (p \wedge q) \Rightarrow q$$

Disjunctive Simplification

$$(p \vee q) \wedge \neg p \Rightarrow q \qquad (p \vee q) \wedge \neg q \Rightarrow p$$

Chain Rule

$$(p \to q) \wedge (q \to r) \Rightarrow (p \to r)$$

Resolution

$$(\neg p \vee r) \wedge (p \vee q) \Rightarrow (q \vee r)$$

Conditional Equivalence

$$p \to q \Leftrightarrow \neg p \vee q$$

Biconditional Equivalences

$$\begin{aligned}(p \leftrightarrow q) &\Leftrightarrow (p \to q) \wedge (q \to p)\\ &\Leftrightarrow (p \wedge q) \vee (\neg q \wedge \neg q)\end{aligned}$$

Contrapositive

$$p \to q \Leftrightarrow \neg q \to \neg p$$

Question 1

(a)

(reason = any correct, relevant statement)

(i) True **(iii)** True **(v)** False

(ii) False **(iv)** True **(vi)** True

6 marks, = 1 mark per correct item

(b)

PROPOSITIONS

- $R =$ "*It is raining*"
- $C =$ "*It is cold*"
- $J =$ "*John is wearing a coat*"

ARGUMENT

prop 1	"*If it is raining, it is not cold*"	$R \to \neg C$
prop 2	"*If is not raining, John is not wearing a coat*"	$\neg R \to \neg J$
prop 3	"*It is cold*"	C
$\therefore$	"*John is not wearing a coat*"	$\neg J$

EXPRESSION

$$(R \to \neg C) \wedge (\neg R \to \neg J) \wedge (C) \to \neg J$$

VIA TRUTH TABLE

R	C	J	$R \to \neg C$	$\neg R \to \neg J$	C	$\neg J$	argument
F	**F**	**F**	**T**	**T**	**F**	**T**	**T**
F	**F**	**T**	**T**	**F**	**F**	**F**	**T**
F	**T**	**F**	**T**	**T**	**T**	**T**	**T**
F	**T**	**T**	**T**	**F**	**T**	**F**	**T**
T	**F**	**F**	**T**	**T**	**F**	**T**	**T**
T	**F**	**T**	**T**	**T**	**F**	**F**	**T**
T	**T**	**F**	**F**	**T**	**T**	**T**	**T**
T	**T**	**T**	**F**	**T**	**T**	**F**	**T**

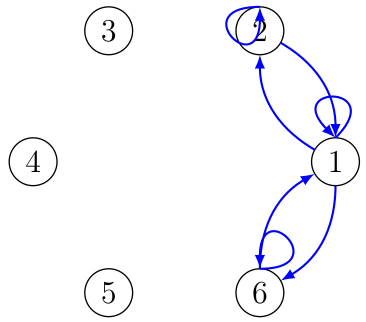
VIA FORMAL ARGUMENT

step 1	$R \to \neg C$	(prop 1)
step 2	$C \to \neg R$	(contrapositive)
step 3	C	(prop 3)
step 4	$\neg R$	(step 3 and step 2)
step 5	$\neg R \to \neg J$	(prop 2)
conclusion	$\neg J$	(step 4 and step 5)

6 marks = 3 for expression + 3 for proof

(c)

(i) *Represent R using a digraph.*



(ii) *Is R reflexive? symmetric? transitive?*

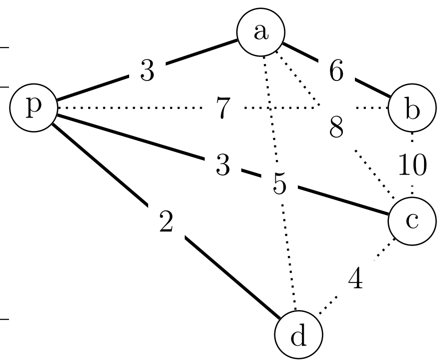
(iii) *Is R an equivalence relation? and if yes, what the resulting equivalence classes?*

7 marks = 2 + 3 (=1+1+1) + 2
- **Need to show difference between Venn diagram and digraph representation.**

(d)

Applying Kruskal's algorithm to the preceding problem we have

Link	Cost	Decision
(p, d)	2	add to tree
(p, a)	3	add to tree
(p, c)	3	add to tree
(d, c)	4	reject as forms circuit $(p, 5, c, p)$
(d, a)	5	reject as forms circuit $(p, 5, a, p)$
(a, b)	6	add to tree



Resulting in a total cost of

$$2 + 3 + 4 + 6 = 15 \text{ million euro}$$

6 marks = 5 algorithm + 1 interpretation
- **Need to show algorithm implementation — including rejecting of edges**

Question 2

(a)

(bookwork)

7 marks = 2 + 2 + 3

- **Need matching to known relation properties (injective, surjective, bijective, function, order preserving, identity, etc) not just translate code**

(b)

(i) *Construct a logical expression to represent this circuit.*

$$(y \wedge \neg x) \vee (\neg y \wedge x)$$

(ii) *Is there an input case for which output is on?*

x	y	$y \wedge \neg x$	$\neg y \wedge x$	$(y \wedge \neg x) \vee (\neg y \wedge x)$
F	**F**	**F**	**F**	**F**
F	**T**	**T**	**F**	**T**
T	**F**	**F**	**T**	**T**
T	**T**	**F**	**F**	**F**

Output is on when exactly one input is one.

(iii) *Hence, construct a logical circuit or equivalent expression, containing three inputs that will have output on when exactly one input is on.*

Define logical operator $x \uparrow y = (y \wedge \neg x) \vee (\neg y \wedge x)$ then required expression is

$$x \uparrow y \uparrow z$$

Note $x \uparrow y$ is the exclusive or operator.

10 marks = 4 + 3 + 3

(c)

CONSTRUCT CONTRAPOSITIVE STATEMENT

Let n be a number greater than 3. Then we wish to prove statement

$$\text{If } \underbrace{n \text{ is a prime number}}_{p}, \text{ then } \underbrace{n \text{ is either one more or one less than a multiple of 6}}_{q}$$

This is logical equivalent to its contrapositive

$$\text{If } \underbrace{n \text{ is either neither one more or one less than a multiple of 6}}_{\neg q}, \text{ then } \underbrace{n \text{ is not a prime number}}_{\neg p}$$

PROVE USING CASES

We have cases based on the remainder of n divided by 6

Case $n \bmod 6 = 0$**:**

$$n \bmod 6 = 0 \iff \exists k \in \mathbb{N}, \quad n = 6k \implies n \text{ is not prime}$$

Case $n \bmod 6 = 1$**:** Not applicable (excluded by premise)

Case $n \bmod 6 = 2$**:**

$$n \bmod 6 = 2 \iff \exists k \in \mathbb{N}, \quad n = 6k + 2 = 2(3k + 1) \implies n \text{ is not prime}$$

Case $n \bmod 6 = 3$**:**

$$n \bmod 6 = 3 \iff \exists k \in \mathbb{N}, \quad n = 6k + 3 = 3(2k + 1) \implies n \text{ is not prime}$$

Case $n \bmod 6 = 4$**:**

$$n \bmod 6 = 4 \iff \exists k \in \mathbb{N}, \quad n = 6k + 4 = 2(3k + 2) \implies n \text{ is not prime}$$

Case $n \bmod 6 = 5$**:** Not applicable (excluded by premise)

Hence, regardless of case, we have that n is not prime. This proves the contrapositive (and original) statement above.

8 marks = 3 contrapositive + 3 separation by cases + 1 conclusion.
Or comparable proof
Partial credit for relevent sample cases

Question 3

(a)

(i) *Construct a recursive definition for the sequence.*

$$a_n = a_{n-1} + 4 \text{ with } a_1 = 5$$

(ii) *Construct a closed formula for the nth term of the sequence.*

$$a_n = 5 + 4(n-1)$$

(iii) *Is 2017 a term in the sequence? Explain.*

Yes, since $2017 = 5 + 4(504 - 1)$ (so $a_{504} = 2017$)

4 marks = 1 + 1 + 2

- **Iterative is not same as recursive formula**

(b)

(i) Since the matrix contains entries other than 1s and 0s, G is not simple. For example, there are 2 edges from vertex 1 to vertex 4.

(ii) The sum of the entries in any row is the degree of the vertex corresponding to that row. The degree sequence is therefore $(2, 2, 3, 3, 4)$.

(iii) The sum of the degrees is 14, and so G has 7 edges.

6 marks = 2 + 2 + 2

(c)

(i) *How many subsets are there of cardinality 4?*

$\binom{6}{4} = 15$ subsets.

(ii) *How many subsets of cardinality 4 have $\{2, 3, 5\}$ as a subset?*

$\binom{3}{1} = 3$ subsets. We need to select 1 of the 3 remaining elements to be in the subset.

(iii) *How many subsets of cardinality 4 contain at least one odd number?*

$\binom{6}{4} = 15$ subsets. All subsets of cardinality 4 must contain at least one odd number.

(iv) *How many subsets of cardinality 4 contain exactly one even number?*

$\binom{3}{1} = 3$ subsets. Select 1 of the 3 even numbers. The remaining three odd numbers of S must all be in the set.

8 marks = 2 + 2 + 2 + 2

(d)

(i) *Squares*

6 squares. Once you pick a dot for the top left corner, the other three dots are determined.

(ii) *Quadrilateral*

$\binom{7}{2}\binom{7}{2} = 441$ quadrilaterals. We must pick two of the seven dots from the top row and two of the seven dots on the bottom row. However, it does not make a difference which of the two (on each row) we pick first because once these four dots are selected, there is exactly one quadrilateral that they determine.

7 marks = 3 + 4

Question 4

(a)

(i) No, element 2 in target set has two incoming arrows.

(ii) Yes, every element in target set has at least one incoming arrow.

4 marks = 2 + 2

(b)

Reason = any correct, relevant statement

(i) No

(ii) Yes

(iii) No

6 marks = 2 + 2 + 2

(c)

(i)

$$\left[5\right] + \left[8\right] + \left[11\right] = 24$$

(ii)

$$(n = 1)\ \ \left[1\right] = 1$$
$$(n = 2)\ \ \left[1\right] + \left[3\right] = 4$$
$$(n = 3)\ \ \left[1\right] + \left[3\right] + \left[5\right] = 9$$
$$(n = 4)\ \ \left[1\right] + \left[3\right] + \left[5\right] + \left[7\right] = 16$$

8 marks = 3 + 5

(d)

Proof Outline: Two cases, apply the Pigeon-Hole principle...

Given a graph with n vertices then consider two separate cases:

- graph is connected so vertices have degree in range $[1, ..., n-1]$.
- graph is not connected so vertices have degree in range $[0, ..., n-2]$.

and think about what happens if you select n numbers from a set of less than n numbers — at least one number must be repeated.

7 marks = 3 separation by cases + 3 pigeonhole principle + 1 conclusion
Or comparable proof
Partial credit for relevent sample cases