WIT

BSc in Applied Computing
BSc in Computer Forensics
BSc in Entertainment Systems
BSc in the Internet of Things

Examination:

Discrete Mathematics
SEMESTER — 1

REPEAT (Autumn 2018)   DURATION: 2 Hours

Internal Examinar: Dr Denis Flynn
Dr Kieran Murphy

Date:

Time:
Venue:

External Examinar:

Instructions to Candidates

1. Answer all Questions.
2. Graph Paper and Statistical Tables Required.
3. Exam Paper (4 pages) and Formula Sheet (1 page)

Waterford Institute of Technology

Question 1

(a) Translate into symbols each of the following. Use $E(x)$ for "x is even" and $O(x)$ for "x is odd."

(i) No integer is both even and odd.

(ii) One more than any even integer is an odd integer.

(iii) There is prime number that is even.

(iv) Between any two real numbers there is a third number.

(v) There is no integer between an integer and one more than that integer.

(10 marks)

(b) After gym class you are tasked with putting the 14 identical basketballs away into 5 bins.

(i) How many ways can you do this if there are no restrictions?

(ii) How many ways can you do this if each bin must contain at least one basketball?

(5 marks)

(c) State, with an explanation, which of the following sequences are the degree sequences of a simple graph. For those sequence that are degree sequences, draw a simple graph with that degree sequence.

(i) $(5, 5, 4, 4, 3, 2, 2, 1, 1)$

(ii) $(6, 6, 6, 5, 3, 2, 2, 2)$

(iii) $(7, 6, 5, 4, 3, 2, 1)$

(6 marks)

(d) A combination lock consists of a dial with 40 numbers on it. To open the lock, you turn the dial to the right until you reach a first number, then to the left until you get to second number, then to the right again to the third number. The numbers must be distinct. How many different combinations are possible? **(4 marks)**

(Total 25 marks)

Question 2

(a) Suppose that U is an infinite universal set, and A and B are infinite subsets of U. Answer the following questions with a brief explanation.

(i) Must $\overline{A}$ be finite?

(ii) Must $A \cup B$ infinite?

(iii) Must $A \cap B$ be infinite?

(5 marks)

(b) Evaluate the following series:

(i) $\displaystyle\sum_{i=1}^{3}(2+3i)$ **(ii)** $\displaystyle\sum_{i=-2}^{1} i^2$

(4 marks)

(c) How many shortest lattice paths start at (3,3) and

(i) end at (10,10)?

(ii) end at (10,10) and pass through (5,7)?

(iii) end at (10,10) and avoid (5,7)?

(6 marks)

(d) Consider the following idealised situation of super computer designed to compute two types of tasks: the first take exactly 3 minutes, and the second takes exactly 5 minutes.
Given the cost in developing the super computer, every effort is make to ensure that the computer achieves maximum utilisation. The standard approach is to run a scheduling process continuously to reorder tasks with the aim of filling any periods (called windows) of inactivity. The role of this scheduling process is if given a window of n minutes determine how many of each type of task can be computed within the n minute window. For example, a window of 8 minutes can be utilised by completing one 3-minute task and one 5-minute task, while a window of 9 minutes can be utilised by completing three 3-minute tasks.
Show, using induction, that it is possible for the scheduling process to utilise fully a window of length n minutes for any integer $n \geq 8$.

(10 marks)

(Total 25 marks)

Question 3

(a) Let $f : X \to Y$ and $g : Y \to Z$ be functions, and consider the composition function $g \circ f$ defined by mapping $x \mapsto g(f(x))$.

(i) If f and g are both injective, must $g \circ f$ be injective? Explain.

(ii) If f and g are both surjective, must $g \circ f$ be surjective? Explain.

(iii) Suppose $g \circ f$ is injective. What, if anything, can you say about f and g? Explain.

(iv) Suppose $g \circ f$ is surjective. What, if anything, can you say about f and g? Explain.

(8 marks)

(b) Let $A = \{1, 2, 3, 4, 5\}$. Construct functions on A, if they exist that have the properties specified below. Justify your answer.

(i) A function that is one-to-one and onto.

(ii) A function that is neither one-to-one nor onto

(iii) A function that is one-to-one but not onto.

(iv) A function that is onto but not one-to-one.

(4 marks)

(c) Using the digits 2 through 8, find the number of different 5-digit numbers such that:

(i) Digits cannot be repeated and must be written in increasing order. For example, 23678 is okay, but 32678 is not.

(ii) Digits *can* be repeated and must be written in *non-decreasing* order. For example, 24448 is okay, but 24484 is not.

(6 marks)

(d) Let $S = \{1, 2, 3, 4, 5, 6\}$

(i) How many subsets are there of cardinality 4?

(ii) How many subsets of cardinality 4 have $\{2, 3, 5\}$ as a subset?

(iii) How many subsets of cardinality 4 contain at least one odd number?

(iv) How many subsets of cardinality 4 contain exactly one even number?

(7 marks)

(Total 25 marks)

Question 4

(a) Evaluate the following:

(i) $\prod_{i=1}^{3} i^2$ **(ii)** $\prod_{i=1}^{3}(2i+1)$

(4 marks)

(b) An Eulerian graph is **randomly traceable** from a vertex, v, if whenever we start from v and traverse the graph in an arbitrary way never using any edge twice, we eventually obtain an Eulerian trail.

(i) Give an example of an Eulerian graph that is not randomly traceable.

(ii) Why might a randomly traceable graph be suitable for the layout of an exhibition?

(7 marks)

(c) Apply Kruskal's algorithm to determine a minimum cost spanning tree for the graph with the following cost matrix. How many such trees are there?

	A	B	C	D	E	F	G	H
A	0	12	0	14	11	0	17	8
B	12	0	9	0	12	15	10	9
C	0	9	0	18	14	31	0	9
D	14	0	18	0	0	6	23	14
E	11	12	14	0	0	15	16	0
F	0	15	31	6	15	0	8	16
G	17	10	0	23	16	8	0	22
H	8	9	9	14	0	16	22	0

(8 marks)

(d) How many derangements are there of 4 elements? **(6 marks)**

(Total 25 marks)

Laws of Logic

Logical Connective	Symbol	Python Operator	Precedence	Logic Gate
Negation (Not)	$\neg$	**not**	Highest	
Conjunctive (And)	$\wedge$	**and**	Medium	
Disjunctive (Or)	$\vee$	**or**	Lowest	

Basic Rules of Logic

Commutative Laws

$p \vee q \Leftrightarrow q \vee p \qquad p \wedge q \Leftrightarrow q \wedge p$

Associative Laws

$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r) \qquad (p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$

Distributive Laws

$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r) \qquad p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

Identity Laws

$p \vee \mathbf{F} \Leftrightarrow p \qquad p \wedge \mathbf{T} \Leftrightarrow p$

Negation Laws

$p \wedge (\neg p) \Leftrightarrow \mathbf{F} \qquad p \vee (\neg p) \Leftrightarrow \mathbf{T}$

Idempotent Laws

$p \vee p \Leftrightarrow p \qquad p \wedge p \Leftrightarrow p$

Null Laws

$p \wedge \mathbf{F} \Leftrightarrow \mathbf{F} \qquad p \vee \mathbf{T} \Leftrightarrow \mathbf{T}$

Absorption Laws

$p \wedge (p \vee q) \Leftrightarrow p \qquad p \vee (p \wedge q) \Leftrightarrow p$

DeMorgan's Laws

$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q \qquad \neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$

Involution Law

$\neg(\neg p) \Leftrightarrow p$

Implications and Equivalences

Detachment (Modus Ponens)

$(p \rightarrow q) \wedge p \Rightarrow q$

Indirect Reasoning (Modus Tollens)

$(p \rightarrow q) \wedge \neg q \Rightarrow \neg p$

Disjunctive Addition

$p \Rightarrow (p \vee q)$

Conjunctive Simplification

$(p \wedge q) \Rightarrow p \qquad (p \wedge q) \Rightarrow q$

Disjunctive Simplification

$(p \vee q) \wedge \neg p \Rightarrow q \qquad (p \vee q) \wedge \neg q \Rightarrow p$

Chain Rule

$(p \rightarrow q) \wedge (q \rightarrow r) \Rightarrow (p \rightarrow r)$

Resolution

$(\neg p \vee r) \wedge (p \vee q) \Rightarrow (q \vee r)$

Conditional Equivalence

$p \rightarrow q \Leftrightarrow \neg p \vee q$

Biconditional Equivalences

$$\begin{aligned}(p \leftrightarrow q) &\Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)\\ &\Leftrightarrow (p \wedge q) \vee (\neg q \wedge \neg q)\end{aligned}$$

Contrapositive

$p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$

Question 1

(a)

(i) *No integer is both even and odd.*

$$\forall x \in \mathbb{Z}(!(E(x) \land O(x)))$$

(ii) *One more than any even integer is an odd integer.*

$$\forall x \in \mathbb{Z}(E(x) \to O(x+1))$$

(iii) *There is prime number that is even.*

$$\exists x \in \mathbb{N}(E(x) \land x \text{ is prime})$$

(iv) *Between any two reals numbers there is a third number.*

$$\forall a \in R \forall c \exists b(a < c \to a < b < c)$$

(v) *There is no integer between an integer and one more than that integer.*

$$\forall x \in \mathbb{Z} \neg \exists y \in Z(x < y < x+1)$$

(b)

(i) *How many ways can you do this if there are no restrictions?*

$\binom{18}{4}$ ways. Each outcome can be represented by a sequence of 14 stars and 4 bars.

(ii) *How many ways can you do this if each bin must contain at least one basketball?*

$\binom{13}{4}$ ways. First put one ball in each bin. This leaves 9 stars and 4 bars.

(c)

(i) No — sum of degree sequence is odd
(ii) No — three vertices of degree 6 in simple graph of 8 vertices implies the there is at most 2 ver
(iii) $(7, 6, 5, 4, 3, 2, 1)$

(d)

Despite its name, we are not looking for a combination here. The order in which the three numbers appears matters. There are $P(40, 3) = 40 \cdot 39 \cdot 38$ different possibilities for the "combination". This is assuming you cannot repeat any of the numbers (if you could, the answer would be 40^3).

Question 2

(a)

Reason = any correct, relevant statement

(i) No

(ii) Yes

(iii) No

(b)

(i) 24 **(ii)** 6

(c)

(i) *start at (3,3) and end at (10,10)?*

$\binom{14}{7} = 3432$ paths. The paths all have length 14 (7 steps up and 7 steps right), we just select which 7 of those 14 should be up.

(ii) *start at (3,3) and end at (10,10) and pass through (5,7)?*

$\binom{6}{2}\binom{8}{5} = 840$ paths. First travel to (5,7), and then continue on to (10,10).

(iii) *start at (3,3) and end at (10,10) and avoid (5,7)?*

$\binom{14}{7} - \binom{6}{2}\binom{8}{5}$ paths.

Remove all the paths found in preceding question.

(d)

(bookwork)— standard induction proof with the inductive step requiring proof by cases $(3 \times 3 \to 2 \times 5)$ and $(5 \to 2 \times 3)$.

Question 3

(a)

(i) *If f and g are both injective, must $g \circ f$ be injective? Explain.*

Yes, any argument based on at most one incoming arrow.

(ii) *If f and g are both surjective, must $g \circ f$ be surjective? Explain.*

Yes, any argument based on at least one incoming arrow.

(iii) *Suppose $g \circ f$ is injective. What, if anything, can you say about f and g? Explain.*

(iv) *Suppose $g \circ f$ is surjective. What, if anything, can you say about f and g? Explain.*

(b)

(i) Many solutions, either formula or graphical representation expected

(ii) Many solutions, either formula or graphical representation expected

(iii) None exist. Why?

(iv) None exist. Why?

(c)

(i) *Digits cannot be repeated and must be written in increasing order.*

There are $\binom{7}{5}$ numbers. Choose five of the seven digits and once chosen put them in increasing order.

(ii) *Digits* can *be repeated and must be written in* non-decreasing *order.*

This requires stars and bars. Use a star to represent each of the 5 digits in the number, and use their position relative to the bars to say what numeral fills that spot. So we will have 5 stars and 6 bars, giving $\binom{11}{6}$ numbers.

(d)

(i) *How many subsets are there of cardinality 4?*

$$\binom{6}{4} = 15 \text{ subsets.}$$

(ii) *How many subsets of cardinality 4 have $\{2, 3, 5\}$ as a subset?*

$\binom{3}{1} = 3$ subsets. We need to select 1 of the 3 remaining elements to be in the subset.

(iii) *How many subsets of cardinality 4 contain at least one odd number?*

$\binom{6}{4} = 15$ subsets. All subsets of cardinality 4 must contain at least one odd number.

(iv) *How many subsets of cardinality 4 contain exactly one even number?*

$\binom{3}{1} = 3$ subsets. Select 1 of the 3 even numbers. The remaining three odd numbers of S must all be in the set.

Question 4

(a)

(i) 36 **(ii)** 105

(b)

(bookwork)

(c)

Should apply Kruskal's algorithm using the cost matrix, without drawing a diagram of the graph. One MST has edges

$$AE, AH, HB, BC, BG, GF, FD$$

There are 3 minimum cost spanning trees.

(d)

We count all permutations, and subtract those which are not derangements.

$$4! - \left[\binom{4}{1}3! - \binom{4}{2}2! + \binom{4}{3}1! - \binom{4}{4}0!\right] = 24 - 15 = 9.$$