Discrete Mathematics

Topic 01 — Logic

Lecture 03 — Quantifiers

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Outline

- Universal and Existential Quantifiers
- Quantifiers and Negation

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Motivation

Consider the statements below. Decide whether any are equivalent to each other, or whether any imply any others.

1. You can fool some people all of the time.
2. You can fool everyone some of the time.
3. You can always fool some people.
4. Sometimes you can fool everyone.

- The mathematical statements that we will encounter in practice will use the connectives “and”, “or”, “not”, “if-then”, and “iff”.
- They will also use quantifiers. While there are many types of quantifiers in English (e.g., many, few, most, etc.) in mathematics we, for the most part, stick to two quantifiers:
  - “for all” “universal”.
  - “there exists” “existential”

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Examples I

Example 1

"All automobiles have wheels"

This statement makes an assertion about **all** automobiles. It is true, because every automobile does have wheels.

Example 2

"There exists a man who has blue eyes"

This statement is of a different nature. It does not claim that all mem have blue eyes—merely that **there exists at least one** man who does. Since that is true, the statement is true.

Example 3

"All positive real numbers are integers"

This sentence asserts that something is true for all positive real numbers. It is indeed true for some positive numbers, such as 1 and 2 and 193. However, it is false for at least one positive number (such as 1/10 or π), so the statement is false.

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Examples II

Example 4

"The square of any real number is positive"

This assertion is almost true — the only exception is the real number 0 (since $0^2 = 0$) is not positive. But it only takes one exception to falsify a "for all" statement. So the assertion is false.
This example illustrates the principle that

> The negation of a "for all" statement is a "there exists" statement.

Example 5

"There exists a real number which is greater than 5"

In fact there are lots of numbers which are greater than 5; some examples are 7, 42, 2π, and $97/3$. Other numbers, such as 1, 2, and $\pi/6$, are not greater than 5. Since there is at least one number satisfying the assertion, the assertion is true.

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Universal and Existential Quantifiers

Definition 6 (Existential Quantifier)

The existential quantifier is $\exists$ and is read "there exists" or "there is." For example

$$\exists x \ [x < 0]$$

asserts that there is a number less than 0. **True**, say $x = -1$

Definition 7 (Universal Quantifier)

The universal quantifier is $\forall$ and is read "for all" or "every." For example,

$$\forall x \ [x \geq 0]$$

asserts that every number is greater than or equal to 0. **False**

- Whenever we are working with either the existential or universal qualifiers we need to know from what collection x is drawn from.*

*More on this when we do number sets.

Predicate Notation - NB!

Predicates can appear difficult — and to be honest they can be difficult, but the main issue in understanding predicate comes from the fact that this is new notation and is very compact. We will try to always write predicates using the following style, and you need to make sure that you practise using it.

$$\overbrace{\text{specifier}\left[\text{statement}\right]}^{\text{qualifier}} \qquad \text{or} \qquad \overbrace{\text{specifier}\left(\text{domain}\right)\left[\text{statement}\right]}^{\text{qualifier}}$$

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Specifies what **variables** will be used, whether the statement needs to be **True** for just one (existence, $\exists$) or for all (universal, $\forall$) **values**

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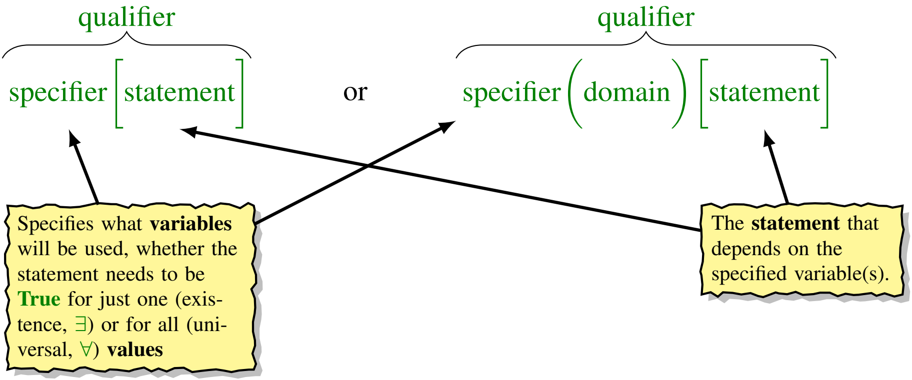

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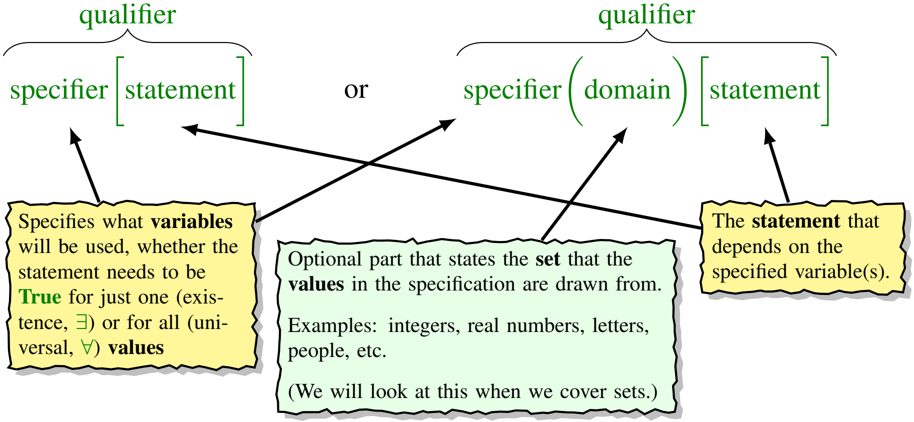

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Translate the following statement into a predicate.

"*Every number is positive*"

Solution

First we will reword the English sentence so that it is closer to the predicate style . . .

"*Every number is positive*"

$\Leftrightarrow$ "*For any number, the number is positive*"

Next we translate into predicate notation . . .

- We need to represent one number, so we will use one symbol, say x.
- The statement "*the number is positive*" can be written as "$x > 0$".

Hence we have predicate . . .

$$\forall x \ [x > 0]$$

Note that this predicate is **False** *since we can find (at least one) value for* x *in which the statement "*$x > 0$*" is* **False**.

$x = -1, -2, \ldots$

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- We need to represent one number, so we will use one symbol, say x.
- Recall, the IFTHEN (or conditional) operator is written using "$\rightarrow$".

Hence we have predicate ...

$$\forall x \; \left[(x > 0) \rightarrow (x > 0)\right]$$

Note that this predicate is always **True** *(hopefully obviously).*

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Translate the following statement into a predicate.

"*Some numbers are positive*"

Solution

First we will reword the English sentence so that it is closer to the predicate style ...

"*Some numbers are positive*"

$\Leftrightarrow$ "*At least one number is positive*"

$\Leftrightarrow$ "*There exists at least one number such that the number is positive*"

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- We need to represent one number, so we will use one symbol, say x.
- Here we are talking about existence, so using "$\exists$".

Hence we have predicate ...

$$\exists x \ [x > 0]$$

Note that this predicate is **True** *since we can find (at least one) value for* x *in which the statement "*$x > 0$*" is* **True**.

$x = 1, 2, \ldots$

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Translate the following statement into a predicate.

"*The average of two positive numbers is positive*"

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First we will reword the English sentence so that it is closer to the predicate style . . .

"*The average of two positive numbers is positive*"

⇔ "*For all two positive numbers, their average is positive*"

⇔ "*For all two numbers,* IF *they are positive* THEN *their average is positive*"

Next we translate into predicate notation . . .

- We need to represent two numbers, so we will use two symbols, x and y.
- The average of x and y is $(x+y)/2$

$$\underbrace{\forall x \forall y}_{\text{For all two numbers,}} \underbrace{\left[(x>0) \wedge (y>0) \rightarrow (x+y)/2\right]}_{\text{IF they are positive THEN their average is positive}}$$

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As with all mathematical statements, we would like to decide whether quantified statements are **True** or **False**. Consider the statement

$$\forall x \exists y \; [y < x]$$

You should read this,

"For all x there exists some y such that y is less than x."

"For every x there is some y such that y is less than x."

Is this statement true?

- The answer depends on what our domain of discourse is: when we say "for all" x, do we mean all positive integers or all real numbers or all elements of some other set?
- Usually this information is implied.
- In discrete mathematics, we almost always quantify over the natural numbers, 0, 1, 2, , so let's take that for our domain of discourse here.

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- The answer depends on what our domain of discourse is: when we say "for all" x, do we mean all positive integers or all real numbers or all elements of some other set?
- Usually this information is implied.
- In discrete mathematics, we almost always quantify over the natural numbers, 0, 1, 2, , so let's take that for our domain of discourse here.

Example II

So our statement, with domain of discourse is

$$\forall x \exists y \ [y < x] \qquad \text{where } x \text{ and } y \text{ are natural numbers } (0, 1, 2, \ldots)$$

- For the statement to be true, we need it to be the case that no matter what natural number we select (for x), there is always some natural number (for y) that is strictly smaller.
- Perhaps we could let y be $x - 1$?
- But here is the problem: what if $x = 0$? Then $y = -1$ and then y is not in our domain of discourse.
- Thus we see that the statement is false because there is a number which is less than or equal to all other numbers. In symbols,

$$\exists x \forall y \ [y \geq x] \qquad \text{where } x \text{ and } y \text{ are natural numbers } (0, 1, 2, \ldots)$$

- To show that the original statement is false, we proved that the negation was true. Notice how the negation and original statement compare.

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Consider the statement

$$\forall x \exists y \; [y > x] \qquad \text{where } x \text{ and } y \text{ are real numbers}$$

Claims that, for any real number x, there is a number y which is greater than it. In the realm of the real numbers this is true. In fact $y = x + 1$ will always do the trick.
Hence this statement is **True**.

On the other hand the statement

$$\exists x \forall y \; [y > x] \qquad \text{where } x \text{ and } y \text{ are real numbers}$$

This has quite a different meaning from the first one. It claims that there is an x which is less than every y. This is obviously false. For instance, x is not less than $y = x - 1$.

$\forall$ and $\exists$ do not commute.

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Consider the statement

$$\forall x \forall y \ \left[x^2 + y^2 \geq 0\right]$$

- This statement is true if the domain of discourse is the real numbers.
- However, it is not true over complex numbers.

While the statement

$$\exists x \exists y \ [x + 2y = 7]$$

is true in the realm of the real numbers. it claims that there exist x and y such that $x + 2y = 7$. Certainly the numbers $x = 3$, $y = 2$ will do the job (although there are many other choices that work as well).

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Outline

Quantifiers and Negation

We can pass the negation symbol over a quantifier, but that causes the quantifier to switch type:

$\neg \forall x \left[P(x)\right]$ is equivalent to $\exists x \left[\neg P(x)\right]$

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- These properties should not be surprising: These statements are effectively saying
  - "if not everything has a property, then something doesn't have that property", and
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Exercises

Question 1

Translate into symbols each of the following. Use $E(x)$ for "x is even" and $O(x)$ for "x is odd."

- No number is both even and odd.
- One more than any even number is an odd number.
- There is prime number that is even.
- Between any two numbers there is a third number.
- There is no number between a number and one more than that number.

Question 2

Translate into English each of the following

a) $\forall x \ \left[E(x) \to E(x+2)\right].$

b) $\forall x \exists y \ \left[\sin(x) = y\right].$

c) $\forall y \exists x \ \left[\sin(x) = y\right].$

d) $\forall x \forall y \ \left[x^3 = y^3 \to x = y\right].$

$\neg \exists x \ [x \text{ expects the Spanish Inquisition}]$