# Discrete Mathematics — Tutorial Sheet 02 — Mathematical Proofs

BSc (H) in App Comp, Ent Sys, Comp Foren, and the IoT

## Direct Proof

**Question 1**
Prove that the sum of two odd numbers is even.

**Question 2**
Prove that the product of two odd numbers is odd.

**Question 3**
Prove the claim "The square of an even natural number is even".

**Question 4**
If $A$ and $B$ are real positive numbers, then prove that

$$\underbrace{\frac{A+B}{2}}_{\text{arithmetic mean}} \geq \underbrace{\sqrt{AB}}_{\text{geometric mean}}$$

Hint: Use fact that $(a-b)^2 = a^2 - 2ab + b^2 \geq 0$.

**Question 5**
Prove the Pythagorean theorem.

**Question 6**
Prove that $x = y$ if and only if $xy = \frac{(x+y)^2}{4}$ . Note, you will need to prove in two "directions" here: the "if" and the "only if" part.

## Proof by Cases

**Question 7**
Prove that for any integer $n$, the number $(n^3 - n)$ is even.

**Question 8**
Prove that every prime number greater than 3 is either one more or one less than a multiple of 6.
Hint. Prove the contrapositive by cases.

**Question 9**
Let $a, b, c, d$ be integers. If $a > c$ and $b > c$, then $\max(a, b) - c$ is always positive.

Proof by Contradiction

**Question 10**

Prove that a triangle cannot have more than one right angle.

**(a)** Prove that the $\sqrt{2}$ is irrational.[1]

**(b)** Prove that $\log_2(3)$ is irrational.

**(c)** Let $n$ be an integer. If $3n + 2$ is odd, then prove that $n$ is odd.

**(d)** Prove that there are an infinite number of primes.[2]

**(e)** Prove that there are no integers $x$ and $y$ such that $x^2 = 4y + 2$.

**(f)** The Pigeonhole Principle: If more than $n$ pigeons fly into $n$ pigeon holes, then at least one pigeon hole will contain at least two pigeons. Prove this.

Proof by Construction

**Question 11**

Prove that $x^n$ can be computed using only $\log_2(n)$ multiplications when $n$ is a power of 2.

This is a special case of the Montgomery algorithm for computing large integer power quickly — a big deal in cryptography!

**(a)** Prove that the sum of the first $n$ positive integers equals $n(n + 1)/2$

Proof by Induction

**Question 12**

For all integers $n$, prove that $n^2 + 5n + 6$ is even.

**Question 13**

Prove that the sum of the first $n$ positive integers equals $n(n + 1)/2$

**Question 14**

Prove for integer $n \geq 4$, that $3^n > 2n^2 + 3n$.

**Question 15**

Consider the following idealised situation of super computer designed to compute two types of tasks: the first take exactly 3 minutes, and the second takes exactly 5 minutes.

Given the cost in developing the super computer, every effort is make to ensure that the computer achieves maximum utilisation. The standard approach is to run a scheduling process continuously to reorder tasks with the aim of filling any periods (called windows) of inactivity. The role of this scheduling process is if given a window of $n$ minutes determine how many of each type of task can be computed within the $n$ minute window. For example, a window of 8 minutes can be utilised by completing one 3-minute task and one 5-minute task, while a window of 9 minutes can be utilised by completing three 3-minute tasks.

Show, using induction, that it is possible for the scheduling process to utilise fully a window of length $n$ minutes for any integer $n \geq 8$.

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[1] "irrational"= "not rational". A **rational** number is a number that can be expressed as quotient of two integers $p$ and $p$ which don't have a common factor.

[2] A **prime** is an integer greater than one with exactly two divisors.