Discrete Mathematics

Topic 04 — Relations and Functions

Lecture 05 — Library of Functions

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Outline

- Graphical representation on the plane
- Preview of useful functions

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Motivation

To date we have covered two graphical representations of functions (relations):

- Venn diagram — two sets representing the source and target with arrow(s) going from elements in the source to elements in the target if the pair are related.
- Diagraph — Applicable when the source and target are identical. Each node represents an element in the source, with directed arrow between nodes if they are related.

These representations are effective when dealing with discrete, finite sets. However, when dealing with continuous or infinite sets it is usually more informative if we use graphical representations based on representing pairs in the function (relation) by points on the 2D Cartesian plane:

- The representation of functions on the 2D Cartesian plane is a major part of next semester's module, *Calculus*.
- So for now we will just demonstrate it via a few examples.

Example — Function on a Discrete, Finite Set

Consider the function f on set $A = \{-1, 0, 1, 2\}$ with rule $x \mapsto -\frac{x^3}{3} + x^2 + \frac{x}{3}$. This function can be represented using

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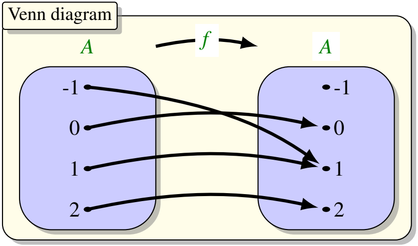

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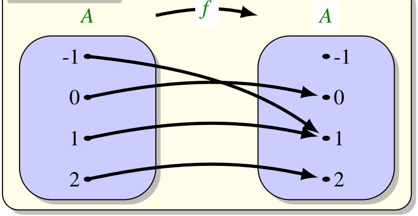
Venn diagram



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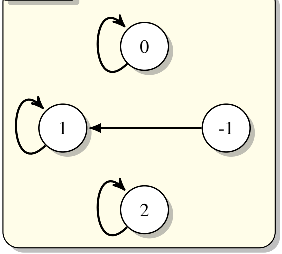

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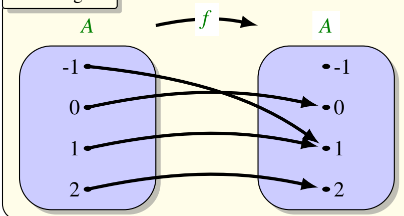

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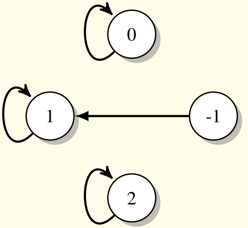
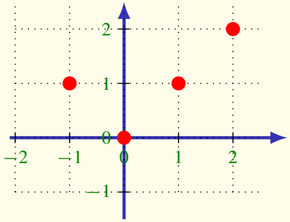


2D Cartesian Graph



Example — Function on a Discrete Infinite Set

Consider the function on $A = \mathbb{N} \setminus \{0\}$ defined by

$$f : A \to A : x \mapsto \text{number of factors of } x$$

- Both representations are only a finite sample of an infinite set — difficult, if not impossible, to be representative.
- Note, in 2D Cartesian graph, it is easy to spot the primes.

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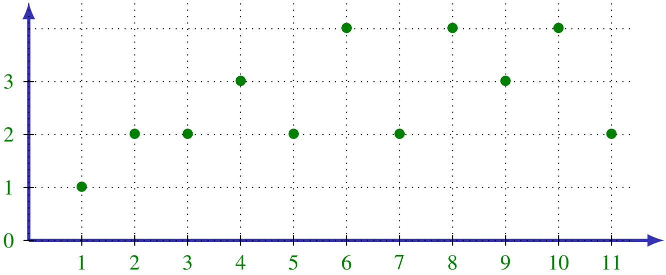

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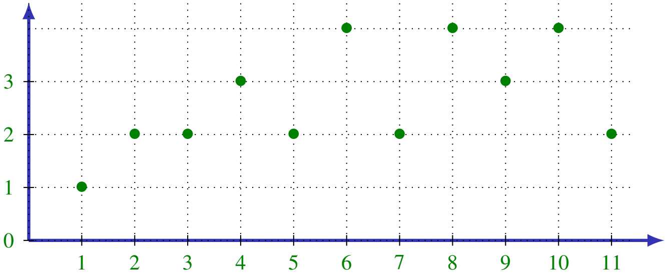

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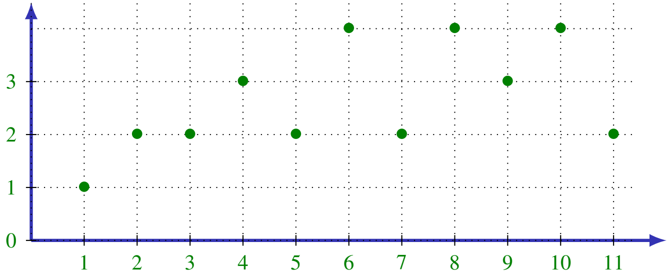

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Example — Function on a Continuous Infinite Set

Consider the function on $\mathbb{R}$ defined by

$$f : \mathbb{R} \to R : x \mapsto \lfloor x \rfloor$$

where $\lfloor x \rfloor$ represents the floor function*

- Solid dot means end point is included, while empty dot means end point is excluded.

*returns the greatest integer that is less than or equal to the input.

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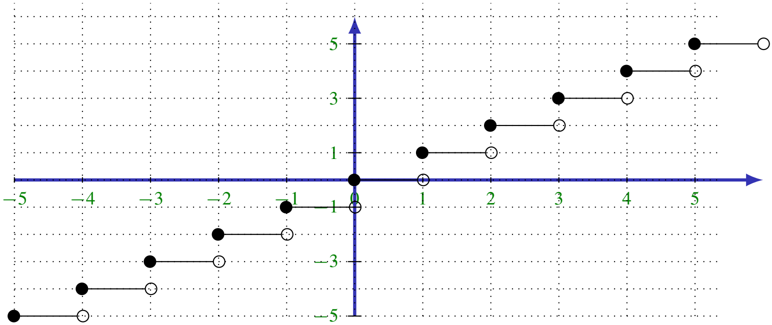

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Outline

Motivation

While our focus in this module is discrete mathematics, and so we are mainly interested in functions on discrete sets. The functions over real numbers that you covered in at second level are also important. For example,

- The running time of algorithms (number of steps required) is usually written as a polynomial, a log, or an exponential.
- The Heaviside is used to model on/off switches.
- The trigonometric functions are used in 3D graphics.
- ...

These functions are cover properly in next semester *Calculus* module. Here, in the following slides, I have a very quick and superficial reminder of some of the more important functions.

Library of Functions

- **Functions**
  - Absolute value function and friends — $|x|$, $\mathrm{sgn}(x)$, $\mathrm{H}(x)$
  - Floor and Ceiling — $\lfloor x \rfloor$ and $\lceil x \rceil$
  - Power — functions of the form $f(x) = x^a$ for $a \in \mathbb{R}$
  - Polynomials
    - constant: $f(x) = c$
    - line: $f(x) = mx + c$
    - quadratic: $f(x) = ax^2 + bx + c$
    - cubic: $f(x) = ax^3 + bx^2 + cx + d$
    - $\ldots$
  - Rational — functions of the form $f(x) = p(x)/q(x)$ where $p(x)$ and $q(x)$ are polynomials
  - Trigonometric
    - Fundamental: sin, cos, tan
    - Inverse: asin, acos, atan
    - Reciprocal: csc, sec, cot
  - Exponential and logarithmic — exp, ln, log
  - Hyperbolic — sinh, cosh, tanh

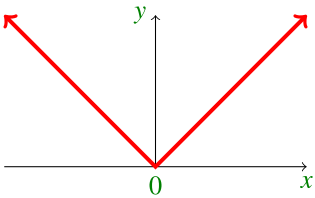
Absolute Value Function and Friends I

Definition 1 (Absolute Value Function)

The *absolute value* function is defined over $\mathbb{R}$ as

$$|x| = \sqrt{x^2} \qquad \text{or equivalently as} \qquad |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

- Satisfies properties
  - $|x| \geq 0$
  - $|x| = 0 \iff x = 0$
  - $|xy| = |x||y|$
  - $|x + y| \leq |x| + |y|$
- Useful for representing distances, e.g., $|x - c| = a$ represents the set of points that are exactly a units distance from c.



Absolute Value Function and Friends II

Definition 2 (Signum function)

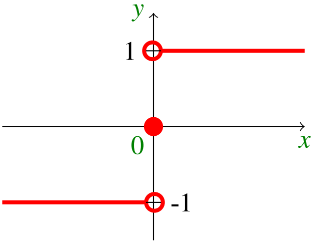
The *signum* function is defined as

$$\operatorname{sgn}(x) \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

- Alternative definition is

$$\operatorname{sgn}(x) = \begin{cases} |x|/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

- Some definitions of $\operatorname{sgn}(x)$ have $\operatorname{sgn}(0)$ as undefined. Again, this is not a big deal as long as everyone knows which definition is used.



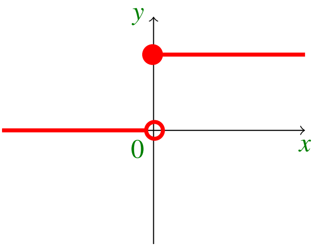
Absolute Value Function and Friends III

Definition 3 (Heaviside functions)

A *Heaviside* function, $H(x)$, is defined as

$$H(x) \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

- The Heaviside function is used to represent a switch turning on at a particular point.
- Alternative definitions using absolute and/or the signum function are possible
- Some definitions set $H(0) = 1/2$. Again, this is not a big deal as long as everyone knows which definition is used.
  The version defined above is called the *right continuous Heaviside* function.



Power Functions I

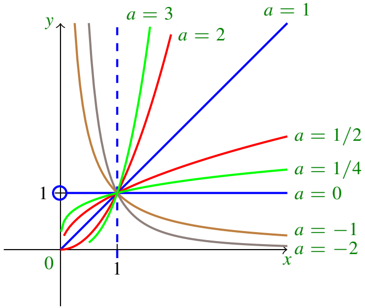
Definition 4 (Power function)

A function of the form

$$f : (0;\infty) \to (0;\infty) : x \mapsto x^a$$

with real constant exponent $a \in \mathbb{R}$, is called a *power function*.

- The domain can be extended for particular values of the exponent. a. For example
  - If $a \in \mathbb{N} \setminus \{0\}$ then $\mathrm{Dom}(f) = \mathbb{R}$. eg $x, x^2, \ldots$
  - If $a \in \mathbb{Z}$ with odd denominator then $\mathrm{Dom}(f) = \mathbb{R} \setminus \{0\}$. eg $x^{\frac{1}{3}}, x^{-\frac{2}{5}}, \ldots$



Polynomials I

Definition 5 (Polynomial Function)

A function $f : \mathbb{R} \to \mathbb{R}$ whose formula is of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

with coefficients $a_k \in \mathbb{R}$, $a_n \neq 0$, is called a *polynomial function* of degree n.

General Properties

1. A polynomial of degree n has up to n roots (zeros/solutions) and up to $n-1$ local minima/maxima.
2. The behaviour for $|x|$ large is determined by the leading order term: $a_n x^n$.
3. The behaviour for $|x|$ small (near zero) is determined by the lowest order non-zero coefficient: a_0 or $a_1 x$ or $a_2 x^2$ etc.

Polynomials are important as they are easy to manipulate, implement on a computer, versatile, and can efficiently model any finite, bounded, smooth curve.

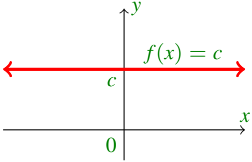
Polynomials II

Case $n = 0$: constant

- If $n = 0$ then

$$f(x) = c$$

for some real constant coefficients c, this is called a *constant function*.



Case $n = 1$: afine

- If $n = 1$ then

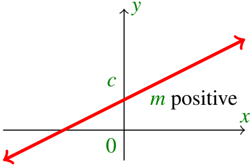
$$f(x) = mx + c$$

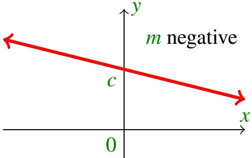
for some real constant coefficients, $m \neq 0$ and c, this is called an *affine function* with slope m, and intercept c.

-
$$\text{slope} = \frac{\text{difference in output}}{\text{difference in input}}$$

- Single root at $x = -c/m$

$f(x) = mx + c$





Polynomials III

Case $n = 2$: quadratic

- If $n = 2$ then

$$f(x) = ax^2 + bx + c$$

for some real constant coefficients, $a \neq 0$, b and c, this is called an *quadratic function*.

- Has *canonical* form

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}$$

(Used to locate local extrema and range.)

- Has roots at

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$