Discrete Mathematics

Topic 05 — Enumeration

Lecture 03 — Combinations and Permutations

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Outline

- Permutations — taking ordered sequences from a collection without repetition.
- Combinations — taking unordered sequences from a collection without repetition.

Outline

Permutations

Definition 1 (Permutations)

A permutation is a (possible) rearrangement of objects.

- For example, there are 6 permutations of the letters *a, b, c*:

  $$abc, \quad acb, \quad bac, \quad bca, \quad cab, \quad cba.$$

  We know that we have them all listed above — there are 3 options for which letter we put first, then 2 options for which letter comes next, which leaves only 1 option for the last letter. The multiplication principle says we multiply $3 \times 2 \times 1$.
- An equivalent definition is: A permutation is any bijective function on a finite set, i.e, source set and target set are the same and have finite number of elements, and the function is one-to-one and onto.

Example

Example 2

How many permutations are there of the letters *a, b, c, d, e, f*?

Solution. We do NOT want to try to list all of these out. However, if we did, we would need to pick a letter to write down first. There are 6 options for that letter. For each option of first letter, there are 5 options for the second letter (we cannot repeat the first letter; we are rearranging letters and only have one of each), and for each of those, there are 4 options for the third, 3 options for the fourth, 2 options for the fifth and finally only 1 option for the last letter.

So there are $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ permutations of the 6 letters.

Permutations of n elements

There are

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

permutations of n (distinct) elements.

Counting Permutations

In general, we can ask how many permutations exist of k objects choosing those objects from a larger collection of n objects where $k \leq n$.

Permutations of k-elements from a collection of n elements

The number of permutations of k elements taken from a set of n (distinct) elements is

$$\mathrm{P}(n,k) = (n)\cdot(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

- The number of different collections of k objects where **order matters** from a collection of n objects is $\mathrm{P}(n,k)$.
- Alternative notation: $\mathrm{P}(n,k) = {}^{n}\mathrm{P}_{k}$.
- $\mathrm{P}(n,k)$ is sometimes called the number of "k-permutations of n elements".
- $\mathrm{P}(n,n) = n!$, i.e., $k = n$

Example

Example 3 (Counting Bijective Functions)

How many functions $f : \{1, 2, \ldots, 8\} \to \{1, 2, \ldots, 8\}$ are *bijective*[a]?

Solution. Each of the 8 elements in the source is mapped to a single distinct element in the target so the number of bijective functions is

$$8 \times 7 \times \cdots \times 1 = 8! = \mathrm{P}(8, 8)$$

[a] Each element in the source is mapped to each element in the target and vice-versa.

Example 4 (Counting injective functions)

How many functions $f : \{1, 2, 3\} \to \{1, 2, 3, 4, 5, 6, 7, 8\}$ are *injective*?

Solution. Note that f cannot be a bijection here. Why?
Using the multiplication principle and using each element in target at most once, the number of injective functions is

$$8 \times 7 \times 6 = \frac{8!}{5!} = \frac{8!}{(8-3)!} = \mathrm{P}(8, 3)$$

Outline

Counting Combinations

If, the order does not matter when drawing k object from a larger collection of n distinct objects we are working with combinations rather than permutations.

Combinations of k-elements from a collection of n elements

The number of combinations of k elements taken from a set of n (distinct) elements is

$$\mathrm{C}(n,k) = \frac{(n)\cdot(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!} = \binom{n}{k} = \frac{\mathrm{P}(n,k)}{k!}$$

- The number of different collections of k objects where **order does not matters** from a collection of n objects is $\mathrm{C}(n,k)$.
- Note the $k!$ in the denominator is to take account of duplicates due to ignoring the order.
- Alternative notation: $\mathrm{C}(n,k) = {}^{n}\mathrm{C}_{k} = \binom{n}{k}$.
- $\mathrm{C}(n,k)$ is sometimes called the number of "k-combinations of n elements".
- $\mathrm{C}(n,n) = \mathrm{C}(n,0) = 1$, i.e., only one way to pick all elements, and only one way to pick zero elements.

Example 5

Example 5

I decide to have a dinner party. Even though, for a mathematician, I'm incredibly popular and have 14 different friends, I only have enough chairs to invite 6 of them.

(a) How many options do I have for which 6 friends to invite?

(b) What if I needed to decide not only which friends to invite but also where to seat them along my long table? How many options do I have then?

Solution.

(a) *How many options do I have for which 6 friends to invite?*
Here I need to pick 6 from a collection of 14 distinct objects. Order does not matter $\implies$ combinations.
This can be done in $\binom{14}{6} = 3003$ ways.

(b) *How many options … to decide … which friends to invite … where to seat them … ?*
Again, I need to pick 6 from a collection of 14 distinct objects. But here order does matter $\implies$ permutations.
So the answer is $P(14, 6) = 2.192.190$.

Review Exercises 1 (Combinations)

Question 1:
A pizza parlour offers 10 toppings.

(a) How many 3-topping pizzas could they put on their menu? Assume double toppings are not allowed.

(b) How many total pizzas are possible, with between zero and ten toppings (but not double toppings) allowed?

(c) The pizza parluor will list the 10 toppings in two equal-sized columns on their menu. How many ways can they arrange the toppings in the left column?

Question 2:
Using the digits 2 through 8, find the number of different 5-digit numbers such that:

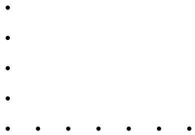
(a) Digits can be used more than once.

(b) Digits cannot be repeated, but can come in any order.

(c) Digits cannot be repeated and must be written in increasing order.

(d) Which of the above counting questions is a combination and which is a permutation? Explain why this makes sense.

Question 3:
How many triangles are there with vertices from the points shown below? Note, we are not allowing degenerate triangles — ones with all three vertices on the same line, but we do allow non-right triangles. Explain why your answer is correct.



Hint. You need exactly two points on either the x- or y-axis, but don't over-count the right triangles.