Discrete Mathematics

Topic 05 — Enumeration

Lecture 04 — Advanced Counting Techniques

Dr Kieran Murphy

Department of Computing and Mathematics,
Waterford IT.
(kmurphy@wit.ie)

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Outline

- Counting problem involving the union of multiple sets.
- Counting problems with identical objects

Outline

Principle of Inclusion and Exclusion

In the addition principle we discussed that given two **disjoint*** sets, A and B, then

$$|A \cup B| = |A| + |B|$$

Q: What happens when A and B have non-zero intersection?
A: When constructing the union, $A \cup B$ from the sets A and B, we can think of adding the elements in the intersection twice, so we have

$$|A \cup B| = |A| + |B| - |A \cap B|$$

This gives us the …

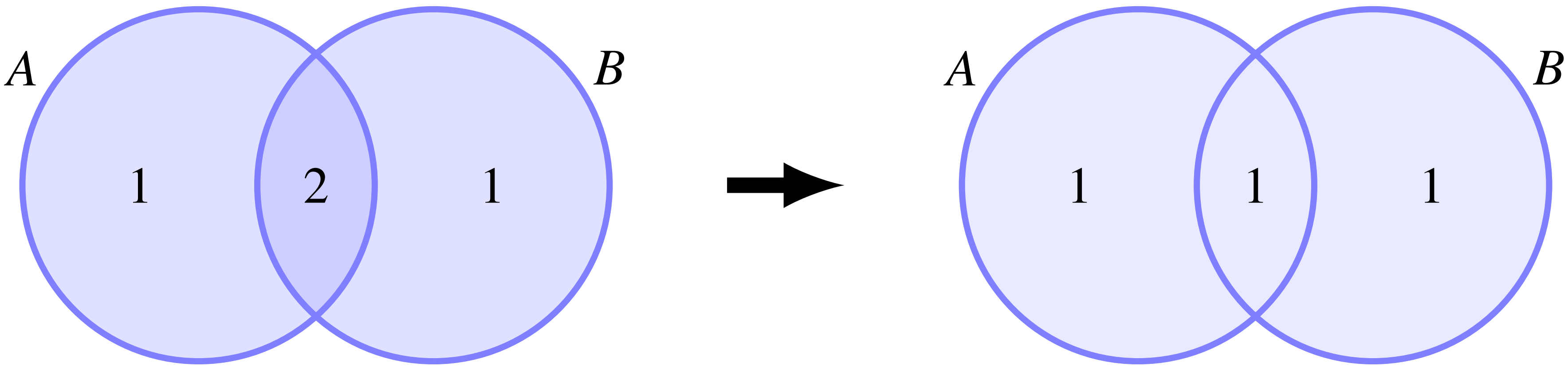
Principle of Inclusion-Exclusion (PIE)

We compute the size of the union of sets in terms of the size of the original sets, by compensating for "repeated addition" (by subtracting) due to the intersections.

* Again disjoint sets have zero intersection.

Principle of Inclusion-Exclusion (PIE) for Two Sets

Venn diagrams are useful in identifying the number of time elements are counted due to intersections of sets. In the following diagram the numbers shown indicate the number of times the elements in that region (set) are counted.
For two sets we need to remove the intersection $A \cap B$.

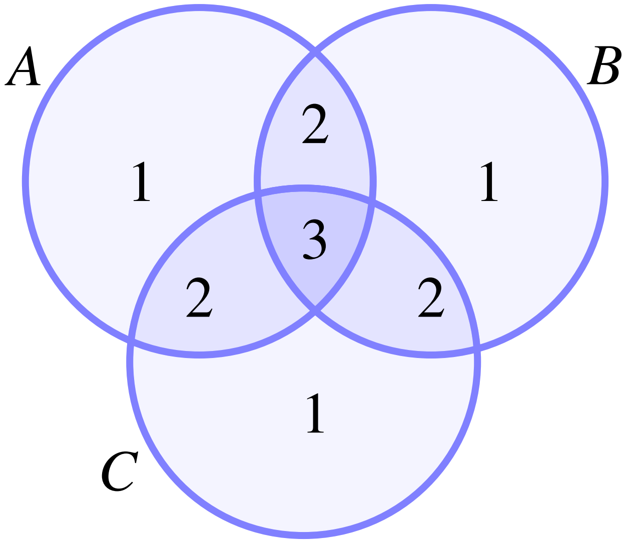


$$|A| + |B|$$

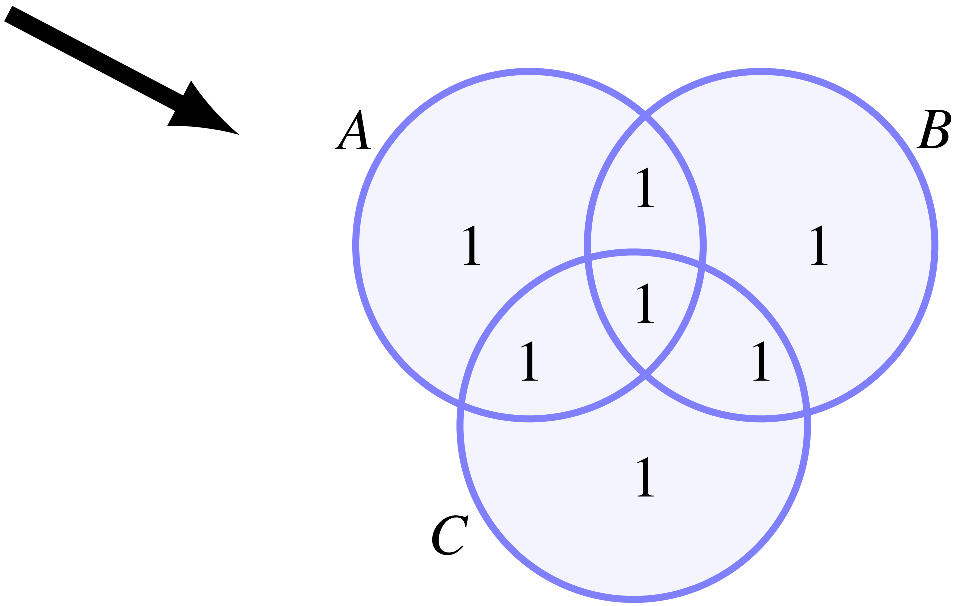
$$|A \cup B| = |A| + |B| - |A \cap B|$$

Principle of Inclusion-Exclusion (PIE) for Three Sets

The same process works for three sets. However, elements in the intersection $A \cap B \cap C$ are counted three times. Removing pairwise intersections $A \cap B$, $A \cap C$ and $B \cap C$ we need to add back the intersection $A \cap B \cap C$ ...



$$|A| + |B| + |C|$$



$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Example

Example 1

A group of college students were asked about their TV watching habits.

Of those surveyed, 28 students watch *The Walking Dead*, 19 watch *The Blacklist*, and 24 watch *Game of Thrones*. Additionally, 16 watch *The Walking Dead* and *The Blacklist*, 14 watch *The Walking Dead* and *Game of Thrones*, and 10 watch *The Blacklist* and *Game of Thrones*. There are 8 students who watch all three shows.

How many students surveyed watched at least one of the shows?

Solution:
Either use the principle of inclusion-exclusion or draw a Venn diagram

$$28 + 19 + 24 - 16 - 14 - 10 + 8 = 39 \text{ students}$$

Outline

Motivation I

Consider the following counting problem:

> You have 7 cookies to give to 4 kids. How many ways can you do this?

- Take a moment to think about how you might solve this problem:
  - You may assume that it is acceptable to give a kid no cookies.
  - Also, the cookies are all identical (while kids are not) and the order in which you give out the cookies does not matter.
- What about the following argument ?
  - For each of the 7 cookies we have a choice with 4 options (which kid gets cookie).
  - So applying the multiplication principle we get

$$\underbrace{4 \times 4 \times \cdots \times 4}_{\text{7 times}} = 4^7$$

However, this argument is wrong since the cookies are identical.

To see why, consider a few possible outcomes: for example, one outcome is assign the first six cookies to kid A, and the seventh cookie to kid B. Another outcome is assign the first cookie to kid B and the six remaining cookies to kid A. Both outcomes are the same, but are included in the 4^7 answer.

Motivation II

Since we can't apply multiplication principle, we will instead try to construct a way to represent an outcome ...

Attempt 1: Number of cookies for each kid

We have four kids A, B, C and D, to specify an outcome we could use a string of four numbers such as

$$3112,$$

which represent the outcome in which the first kid, A, gets 3 cookies, and B gets 1, C gets 1 and D gets 2.

- ✔ The order in the representation is important.
  For example, 1312 *is a different outcome, because then A gets one cookie instead of three cookies, etc..*
- ✔ Each number in the string can be any integer between 0 and 7.
- ✔ But the number of outcomes is not 7^4, since we need the *sum* of the numbers to be 7.
- ✘ Representation is valid but not very helpful in actually determine the count.

Motivation III

Attempt 2: List which kid gets each cookie

Since we have four kids A, B, C and D, we could represent an outcome by writing a string of seven letters — which kid gets each of the 7 cookies. For example

ABAADCD,

represents that the first cookie goes to kid A, the second cookie goes to kid B, and so on.

- ✔ This outcome is identical to 3112 using the previous attempt since A gets 3 cookies, B and C get 1 each and D gets 2.
- ✘ Each of the seven letters in the string can be any of the 4 possible letters (one for each kid), but the number of such strings is not 4^7, because here order does *not* matter.
- ✔ In fact, another way to write the same outcome is

  AAABCDD.

  This will be the preferred representation of the outcome. Since we can write the letters in any order, we might as well write them in *alphabetical* order for the purposes of counting.

Motivation IV

Given the outcome representation

$$\text{AAABCDD}.$$

what is the minimum information we need to represent?

Attempt 3: Stars and bars diagram

Imagine our cookie giving out machine — all we really need to do is say when to switch from one kid/letter to the next.

- In terms of cookies, we need to say after how many cookies do we stop giving cookies to the first kid and start giving cookies to the second kid. And then after how many do we switch to the third kid? And so on.
- So yet another way to represent an outcome is like this:

$$* * * \mid * \mid * \mid * *$$

  where

  - stars "$*$" represent cookies (we have 7 cookies/stars)
  - bars "$|$" represent switching to next kid (we have $3 = 4 - 1$ bars)

Motivation V

So the outcome

$$* * * | * | * | * *$$

represents three cookies go to the first kid, then we switch and give one cookie to the second kid, then switch, one to the third kid, switch, two to the fourth kid.

- Notice that we need 10 symbols — 7 stars and 3 bars — one star for each cookie, and one bar for each switch between kids.
- In particular, we need one fewer bars than there are kids.

So What ?

- To count the number of ways to distribute 7 cookies to 4 kids, all we need to do is count how many *stars and bars* charts there are.
- But a *stars and bars chart* is just a string of symbols, some stars and some bars.
- If instead of stars and bars we would use 0's and 1's, it would just be a bit string. We know how to count those!!

There are $\binom{n+k}{k}$ stars and bars charts with n stars and k bars.

Motivation VI

There are two special cases of *stars and bars* charts that we need to make sure we are interpreting correctly:

- Case 1: Adjacent bars
  The stars and bars chart
  $$*|**||****$$
  represents the outcome in which kid A gets 1 cookie, kid B gets two cookies, kid C gets 0 cookies (no stars before the next bar) and kid D gets the remaining 4 cookies.
- Case 2: Bars at start/end of string
  The stars and bars chart
  $$|******|*|$$
  represents the outcome in which kid A gets 0 cookies (because we switch to kid B before any stars), B gets 6 cookies and kid C gets 1 cookie, and D get nothing,

Finally, we can now answer the question . . .

Example 2

Example 2

You have 7 cookies to give to 4 kids.

(a) How many ways can you do this?

(b) If each kid is to get at least one cookie, then how many ways?

(a) *7 cookies to give to 4 kids . . .*

- Each distribution of the 7 cookies among the 4 kids can be represented as a stars and bars chart with 10 symbols — 7 stars (cookie) and 3 bars (switch to next kid).
- There are $\binom{10}{3}$ stars and bars charts with 10 symbols and 3 bars.

Thus, there are $\binom{10}{3} = 120$ ways to distribute 7 cookies to 4 kids.

(b) *7 cookies to give to 4 kids, with each kid to get at least one . . .*

- Give a cookie to each kid. Then have 3 cookies left with no restriction.
- We have 3 cookies to give to 4 kids. Now have a stars and bars chart with 3 stars and 3 bars.

Thus, there are $\binom{6}{3=20}$ ways to distribute 7 cookies to 4 kids, with each kid getting at least one cookie.

Example 3

Example 3

How many 7 digit phone numbers are there in which the digits are non-increasing? That is, every digit is less than or equal to the previous one.

Solution. We need to decide on 7 digits so we will use 7 stars. The bars will represent a switch from each possible single digit number down the next smaller one. So the phone number 866-5221 is represented by the stars and bars chart

$$| * || * * | * ||| * * | * |$$

There are 10 choices for each digit (0-9) so we must switch between choices 9 times. We have 7 stars and 9 bars, so the total number of phone numbers is

$$\binom{16}{9}.$$

Example 4 1

Example 4

How many integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 13.$$

(An **integer solution** to an equation is a solution in which the unknown must have an integer value.)

(a) where $x_i \geq 0$ for each x_i?

(b) where $x_i > 0$ for each x_i?

(c) where $x_i \geq 2$ for each x_i?

Solution.

This problem is just like giving 13 cookies to 5 kids. We need to say how many of the 13 units go to each of the 5 variables. In other words, we have 13 stars and 4 bars (the bars are like the "+" signs in the equation).

Example 4 II

(a) *Number of solutions where* $x_i \geq 0$ *for each* x_i
If x_i can be 0 or greater, we are in the standard case with no restrictions. So 13 stars and 4 bars can be arranged in $\binom{17}{4}$ ways.

(b) *Number of solutions where* $x_i > 0$ *for each* x_i
Now each variable must be at least 1. So give one unit to each variable to satisfy that restriction.
Now there are 8 stars left, and still 4 bars, so the number of solutions is $\binom{12}{4}$.

(c) *Number of solutions where* $x_i \geq 2$ *for each* x_i
Now each variable must be 2 or greater. So before any counting, give each variable 2 units.
We now have 3 remaining stars and 4 bars, so there are $\binom{7}{4}$ solutions.

Review Exercises 1 (Stars and Bars Technique)

Question 1:
After gym class you are tasked with putting the 14 identical dodgeballs away into 5 bins.

(a) How many ways can you do this if there are no restrictions?

(b) How many ways can you do this if each bin must contain at least one dodgeball?

Question 2:
How many integer solutions are there to the equation $x + y + z = 8$ for which

(a) x, y, and z are all positive?

(b) x, y, and z are all non-negative?

(c) x, y, and z are all greater than -3.

Question 3:
Using the digits 2 through 8, find the number of different 5-digit numbers such that:

(a) Digits cannot be repeated and must be written in increasing order. For example, 23678 is okay, but 32678 is not.

(b) Digits *can* be repeated and must be written in *non-decreasing* order. For example, 24448 is okay, but 24484 is not.