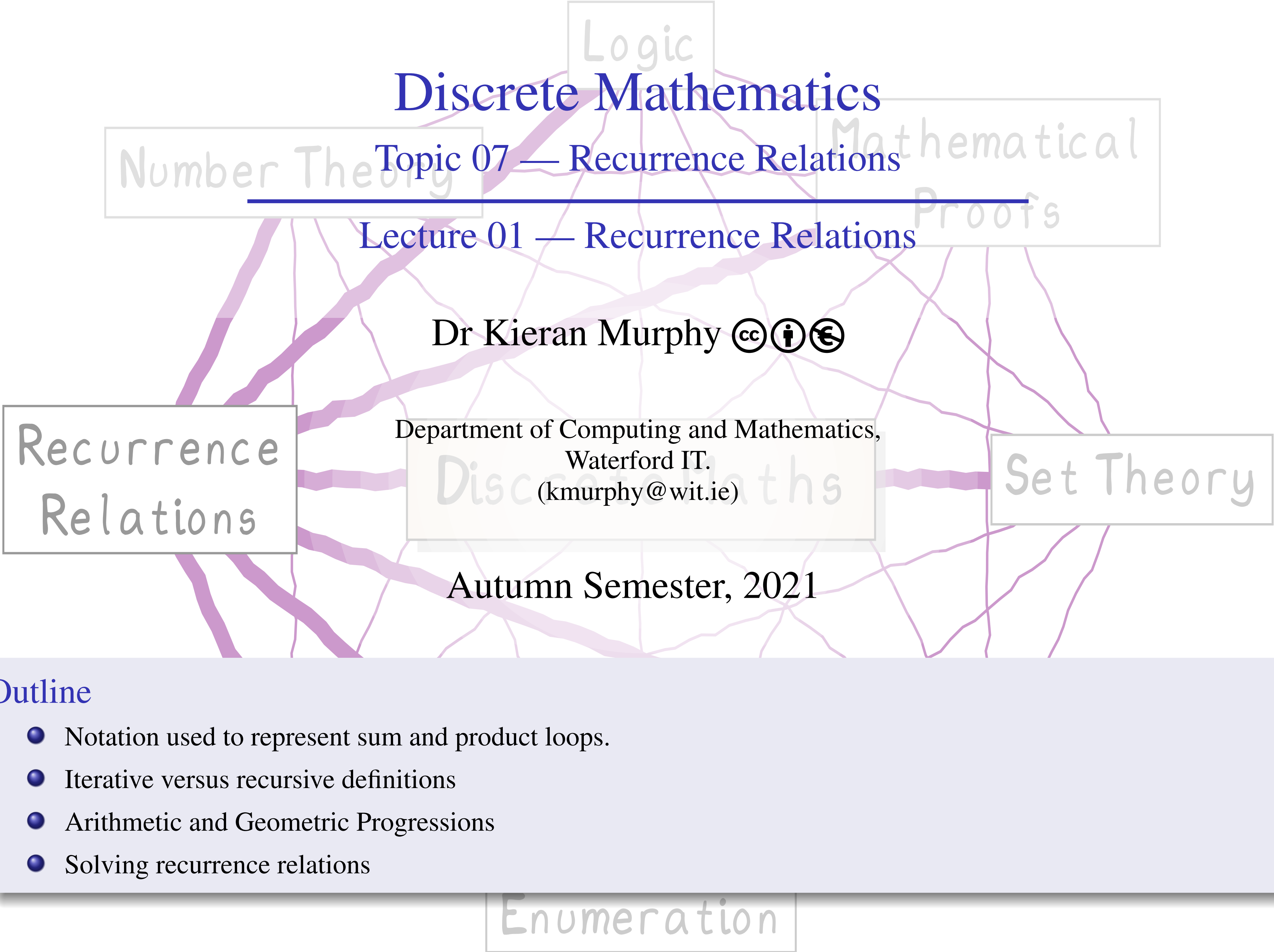

Discrete Mathematics

Topic 07 — Recurrence Relations

Lecture 01 — Recurrence Relations

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Outline

- Notation used to represent sum and product loops.
- Iterative versus recursive definitions
- Arithmetic and Geometric Progressions
- Solving recurrence relations

Outline

Summation Notation

- The $\sum$ operator is used to denote the addition of a, possibly large, number of elements from a sequence/list.
- It can be implemented using a **for** loop in Python/Java/Processing.

Example 1

$$\underbrace{\sum_{k=1}^{10}\left[k^2\right]}_{\substack{\text{“Determine the value of expression within the brackets}\\ \text{as } k\,=\,1,2,3,\ldots,10 \text{ and add all the results.”}}} = \underbrace{1^2}_{k=1} + \underbrace{2^2}_{k=2} + \underbrace{3^2}_{k=3} + \underbrace{4^2}_{k=4} + \cdots + \underbrace{10^2}_{k=10}$$

$$= 1 + 4 + 9 + 16 + 25 + 36 + \cdots + 100$$

$$= 385$$

sum.py

```
result = 0
for k in range(1,11):
    term = k*k
    result += term
print(result)
```

385

sum.py

```
sequence = [k*k for k in range(1,11)]
result = sum(sequence)
print(result)
```

385

Product Notation

- The $\prod$ operator is used to denote the product of a, possibly large, number of elements from a sequence/list.
- It can be implemented using a **for** loop in Python/Java/Processing.

Example 2

$$\underbrace{\prod_{k=1}^{10}\left[k^2\right]}_{} = \underbrace{1^2}_{k=1} \times \underbrace{2^2}_{k=2} \times \underbrace{3^2}_{k=3} \times \underbrace{4^2}_{k=4} \times \cdots \times \underbrace{10^2}_{k=10}$$

"Determine the value of expression within the brackets as $k = 1, 2, 3, \ldots, 10$ and multiply all the results."

$$= 1 \times 4 \times 9 \times 16 \times 25 \times 36 \times \cdots \times 100$$

$$= 13,168,189,440,000$$

product.py

```
result = 1
for k in range(1,11):
    term = k*k
    result *= term
print(result)
```

```
13168189440000
```

There is no product function, similar to the **sum** function in Python.

FYI: Guido van Rossum, vetoed it `https://bugs.python.org/issue1093` and see **reduce** function

Sequence

Informally, a sequence is just an ordered list of numbers. Since the order is important we can label the values in the list, starting with zero, then one and so on. This gives us the formal definition of a sequence

Definition 3 (Sequence)

A sequence is a function from the set of natural numbers, $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$ to a some set A. So we have

$$\begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & & n & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \cdots \\ a_0 & a_1 & a_2 & a_3 & a_4 & & a_n & \end{array}$$

and

- a_n is the image of n, and is called a term/element of the sequence.

- To refer to the *entire* sequence at once, we will write $(a_n)_{n \in \mathbb{N}}$ or $(a_n)_{n \geq 0}$, or if we are being sloppy, just (a_n) (in which case we assume we start the sequence with a_0).
- We might replace the a with another letter, and sometimes we omit a_0, starting with a_1, in which case we would use $(a_n)_{n \geq 1}$ to refer to the entire sequence.
- The numbers in the subscripts are called indices (the plural of index).

Examples of Sequences

- The sequence $a_n = n^2$, where $n = 1, 2, 3, \ldots$ has elements

$$1, 4, 9, 16, 25, 36, 49, \ldots$$

- The sequence $a_n = (-1)^n$, where $n = 0, 1, 2, \ldots$ has elements

$$1, -1, 1, -1, 1, -1, \ldots$$

- The sequence $a_n = 2^n$, where $n = 0, 1, 2, \ldots$ has elements

$$1, 2, 4, 8, 16, 32, 64, 128, \ldots$$

- The Fibonacci sequence has elements

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$$

Closed vs Recursive Formula for Sequences

Since no number of initial terms in a sequence is enough to define a sequence we need to specify a rule for the general term in the sequence — we have two options:

Definition 4 (Closed Formula and Recursive Definition)

- A closed formula for a sequence $(a_n)_{n\in\mathbb{N}}$ is a formula for a_n using a fixed finite number of operations on n.).
- A recursive definition for a sequence $(a_n)_{n\in\mathbb{N}}$ consists of a recurrence relation: an equation relating a term of the sequence to previous terms (terms with smaller index) and an initial/terminal condition.

Example

The Fibonacci sequence $(a_n) = (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 5)$ has closed formula

$$a_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1+\sqrt{5}}{2}\right)^{-n}}{\sqrt{5}}$$

Hard to obtain, easy to use

and recursive formula

$$\underbrace{a_n = a_{n-1} + a_{n-2}}_{\text{recurrence relation}} \quad \text{and} \quad \underbrace{a_0 = 0, \quad a_1 = 1}_{\text{terminal conditions}}$$

Easy to obtain, hard to use

Example

Example 5

Find a_6 in the sequence defined by $a_n = 2a_{n-1} - a_{n-2}$ with $a_0 = 3$ and $a_1 = 4$.

Solution. Using $n = 6$, we know that $a_6 = 2a_5 - a_4$. So to find a_6 we need to find a_5 and a_4.
Similarly, working backwards we have

$$a_5 = 2a_4 - a_3, \quad a_4 = 2a_3 - a_2, \quad a_3 = 2a_2 - a_1 \quad \text{and} \quad a_2 = 2a_1 - a_0,$$

So now knowing a_1 and a_0 we can work forwards again:

$$\begin{aligned} a_0 &= 3 \\ a_1 &= 4 \\ a_2 &= 2 \cdot 4 - 3 = 5 \\ a_3 &= 2 \cdot 5 - 4 = 6 \\ a_4 &= 2 \cdot 6 - 5 = 7 \\ a_5 &= 2 \cdot 7 - 6 = 8 \\ a_6 &= 2 \cdot 8 - 7 = 9. \end{aligned}$$

Note that in this case a closed formula for a_n exists. Namely, $a_n = n + 3$.
A closed formula is easier to use to calculate a general term, but it is often much harder, if not impossible, to derive.

Review Exercises 1 (Definitions and Notation)

Question 1:
Expand the following sums

(a) $\sum_{k=4}^{7} k$ **(b)** $\sum_{k=1}^{5} (k^1 - 1)$ **(c)** $\sum_{n=1}^{4} (10^n)$ **(d)** $\sum_{k=1}^{5} (k^1 - 1)$

Question 2:
Write the following expressions using summation notation

(a) $2 + 4 + 6 + 8 + 10$ **(b)** $1 + 4 + 7 + 10$ **(c)** $\frac{1}{4} + \frac{1}{2} + 1 + 2 + 4$

Question 3:
Expand the following sums

(a) $\prod_{k=-4}^{4} k$ **(b)** $\prod_{k=1}^{4} (k^1 - 1)$ **(c)** $\prod_{k \in S} (-1)^k$ where $S = \{2, 4, 6, 7\}$.

Question 4:
For each of the following sequences, determine a recursive definition.

(a) $2, 4, 6, 10, 16, 26, 42, \ldots$.

(b) $5, 6, 11, 17, 28, 45, 73, \ldots$.

(c) $0, 0, 0, 0, 0, 0, 0, \ldots$.

Question 5:
Show that $a_n = 3 \cdot 2^n + 7 \cdot 5^n$ is a solution to the recurrence relation $a_n = 7a_{n-1} - 10a_{n-2}$. What would the initial conditions need to be for this to be the closed formula for the sequence?

Outline

Arithmetic Progression/Sequence

Definition 6 (Arithmetic Progression/Sequence (AP))

A sequence is called arithmetic if the terms of the sequence differ by a constant. Suppose the initial term (a_0) of the sequence is a and the common difference is d, then we have sequence

$$\underbrace{a,}_{a_0} \xrightarrow{+d} \underbrace{a+d,}_{a_1} \xrightarrow{+d} \underbrace{a+2d,}_{a_2} \xrightarrow{+d} \underbrace{a+3d,}_{a_3} \xrightarrow{+d} \cdots \quad \underbrace{a+nd,}_{a_n} \xrightarrow{+d} \cdots$$

Recursive definition: $a_n = a_{n-1} + d$ with $a_0 = a$.
Closed formula: $a_n = a + dn$.

Example 7

Find recursive definitions and closed formulas for the sequences below. Assume the first term listed is a_0.

1. $2, 5, 8, 11, 14, \ldots$.
2. $50, 43, 36, 29, \ldots$.

Geometric Progression/Sequence

Definition 8 (Geometric Progression/Sequence (GP))

A sequence is called geometric if the ratio between successive terms is constant. Suppose the initial term a_0 is a and the common ratio is r. Then we have, sequence

$$\underbrace{a,}_{a_0} \quad \underbrace{ar,}_{a_1} \quad \underbrace{ar^2,}_{a_2} \quad \underbrace{ar^3,}_{a_3} \quad \dots \quad \underbrace{ar^n,}_{a_n} \quad \dots$$

Recursive definition: $a_n = ra_{n-1}$ with $a_0 = a$.
Closed formula: $a_n = ar^n$.

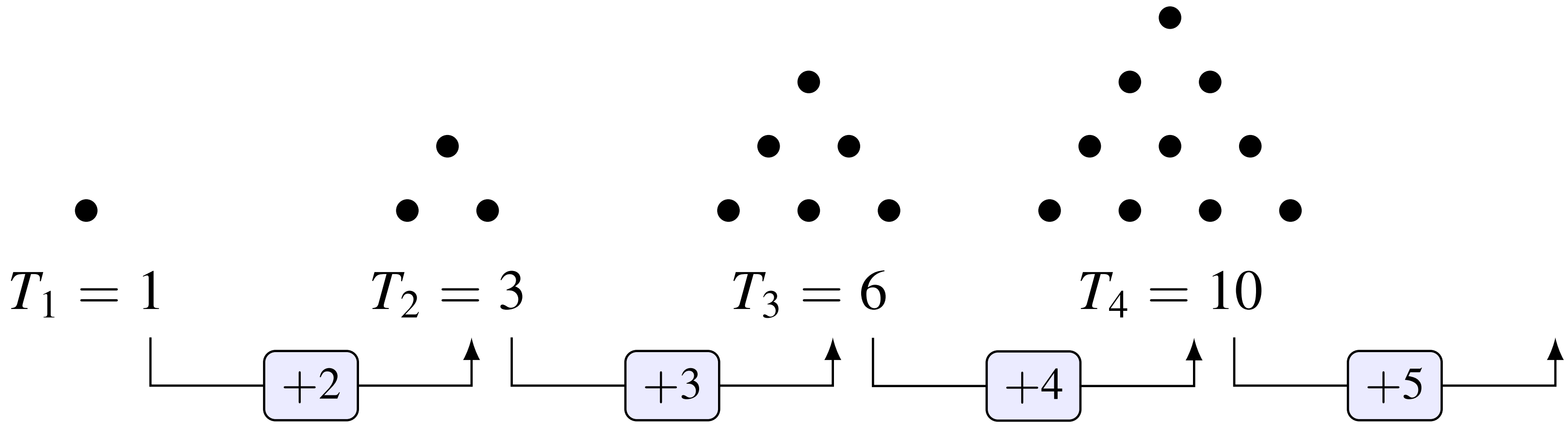
Example 9

Find the recursive and closed formula for the sequences below. Again, the first term listed is a_0.

1. $3, 6, 12, 24, 48, \dots$
2. $27, 9, 3, 1, 1/3, \dots$

Motivation

Look at the sequence $(T_n)_{n\geq 1}$ which starts $1, 3, 6, 10, 15, \ldots$. These are called the triangular numbers since they represent the number of dots in an equilateral triangle (think of how you arrange 10 bowling pins: a row of 4 plus a row of 3 plus a row of 2 and a row of 1).



- Is this sequence arithmetic?
  No, since $3 - 1 = 2$ and $6 - 3 = 3 \neq 2$, so there is no common difference.
- Is the sequence geometric?
  No. $3/1 = 3$ but $6/3 = 2$, so there is no common ratio.
- However, notice that the differences between terms generate an arithmetic sequence: $2, 3, 4, 5, 6, \ldots$. This says that the nth term of the triangular sequence is the sum of the first n terms in the sequence $1, 2, 3, 4, 5, \ldots$, i.e, the triangular sequence is a sequence of partial sums.

Summing Arithmetic Sequences: Reverse and Add I

Example 10

Find the sum: $2 + 5 + 8 + 11 + 14 + \cdots + 470$.

Solution. If we add the first and last terms, we get 472. The second term and second-to-last term also add up to 472. To keep track of everything, we might express this as follows. Call the sum S. Then,

$$\begin{array}{rrcrcrcrcrcrcr} & S = & 2 & + & 5 & + & 8 & + \cdots + & 467 & + & 470 \\ + & S = & 470 & + & 467 & + & 464 & + \cdots + & 5 & + & 2 \\ \hline & 2S = & 472 & + & 472 & + & 472 & + \cdots + & 472 & + & 472 \end{array}$$

Hence, to find $2S$ then we add 472 to itself a number of times. What number? We need to decide how many terms are in the sum. Since the terms form an arithmetic sequence, the nth term in the sum (counting 2 as the 0th term) can be expressed as $2 + 3n$. If $2 + 3n = 470$ then $n = 156$. So n ranges from 0 to 156, giving 157 terms in the sum. This is the number of 472's in the sum for $2S$. Thus

$$2S = 157 \times 472 = 74104 \quad \Longrightarrow \quad S = \frac{74104}{2} = 37052$$

Summing Arithmetic Sequences: Reverse and Add II

The process covered in the previous slide will work for any sum of arithmetic sequences.

STEP 1 Call the sum S.

STEP 2 Reverse and add.

STEP 3 This produces a single number added to itself many times.

STEP 4 Determine the number of times.

STEP 5 Multiply. Divide by 2. Done

Definition 11 (Arithmetic Series)

The sum of the terms of the arithmetic sequence

$$S_n = [a] + [a+d] + [a+2d] + \cdots + [a+nd]$$

is called an arithmetic series and is given by

$$S_n = (n+1)a + \frac{dn(n+1)}{2}$$

Summing Geometric Sequences: Multiply and Subtract I

To find the sum of a geometric sequence, we cannot just reverse and add. Instead we multiply and subtract:

Example 12

What is $3 + 6 + 12 + 24 + \cdots + 12288$?

This terms in the sum are from a geometric progression with initial term, $a_0 = 3$, and common ratio, $r = 2$.

STEP 1 Call the sum S.

STEP 2 Multiply each term by the common ratio, $r = 2$

STEP 3 Subtract, and solve for S.

$$\begin{array}{rrll} S = & 3+ & 6 + 12 + 24 + \cdots + 12288 & \\ 2S = & & 6 + 12 + 24 + \cdots + 12288 & +24576 \\ \hline -S = & 3+ & 0 + 0 + 0 + \cdots + 0 & -24576 \end{array}$$

$$-S = 3 - 24576 \implies S = 24573$$

Summing Geometric Sequences: Multiply and Subtract II

Definition 13 (Geometric Series)

The sum of the terms of the geometric sequence

$$S_n = \left[a\right] + \left[ar\right] + \left[ar^2\right] + \cdots + \left[ar^n\right]$$

is called a geometric series and is given by

$$S_n = \frac{a(1 - r^{n+1})}{1 - r}$$

- In the special case of $-1 < r < 1$ the terms in the geometric sequence tends towards zero fast enough that the sum of the series tends to the finite value

$$S_\infty = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{a}{1 - r}$$

since $r^{n+1} \to 0$ as $n \to \infty$.

Review Exercises 2 (Arithmetic and Geometric Progressions)

Question 1:

Consider the sequence $5, 9, 13, 17, 21, \ldots$ with $a_1 = 5$

(a) Give a recursive definition for the sequence.

(b) Give a closed formula for the nth term of the sequence.

(c) Is 2013 a term in the sequence? Explain.

(d) How many terms does the sequence $5, 9, 13, 17, 21, \ldots, 533$ have?

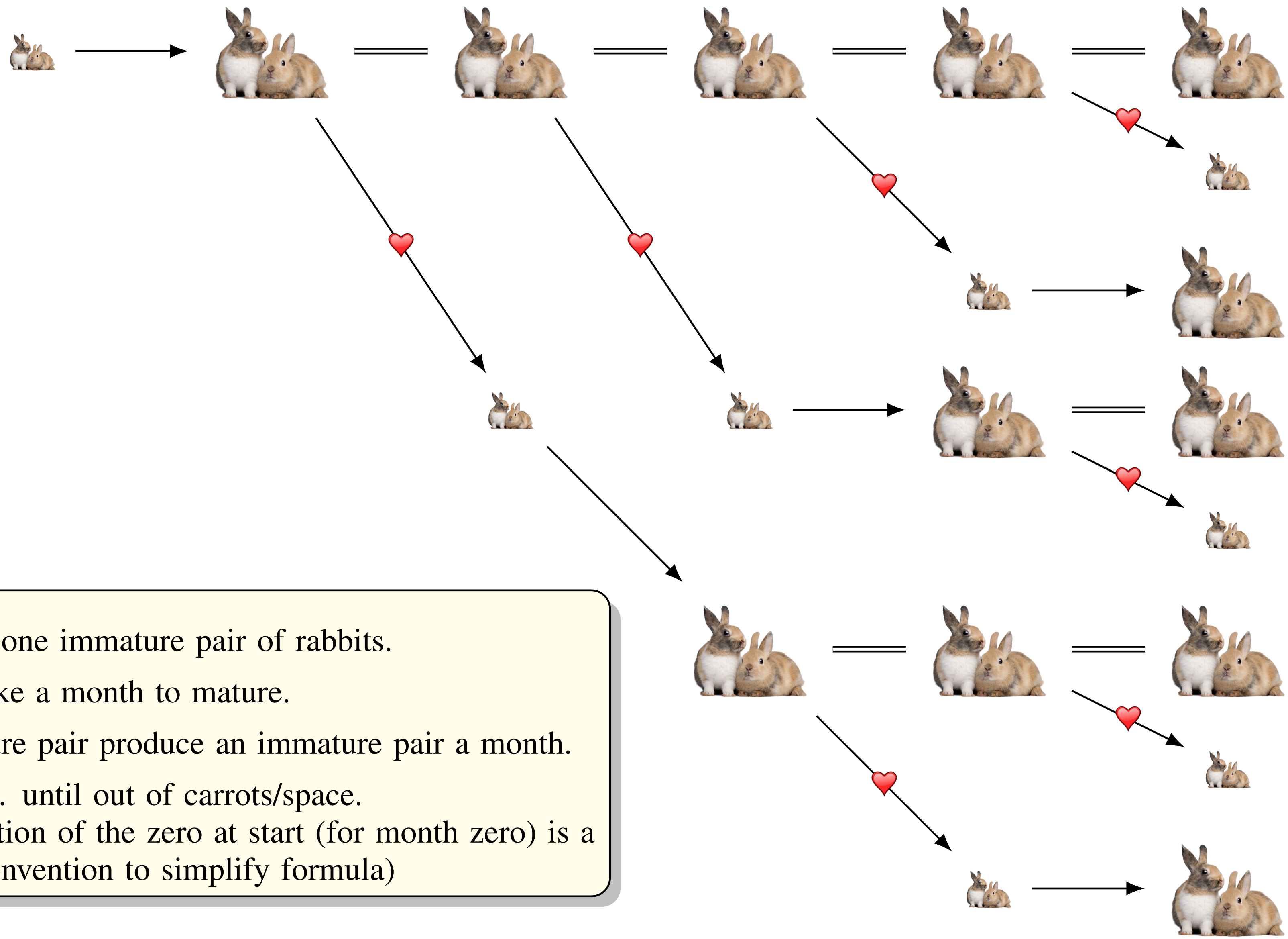
(e) Determine the sum: $5 + 9 + 13 + 17 + 21 + \cdots + 533$. Show your work.

(f) Use what you found above to find b_n, the n^{th} term of $1, 6, 15, 28, 45, \ldots$, where $b_0 = 1$

Outline

Fibonacci Sequence — Rabbit Population

Month	1	2	3	4	5	6
Population (In pairs)	1	1	2	3	5	8



- Start with one immature pair of rabbits.
- Rabbits take a month to mature.
- Each mature pair produce an immature pair a month.
- Repeat … until out of carrots/space.
  (The insertion of the zero at start (for month zero) is a modern convention to simplify formula)

Solving Recurrence Relations

The Fibonacci has recursive definition

$$\underbrace{a_n = a_{n-1} + a_{n-2}}_{\text{recurrence relation}} \qquad \text{and} \qquad \underbrace{a_0 = 0, \quad a_1 = 1}_{\text{terminal conditions}}$$

We would like to obtain a closed formula for a_n. To do this we will study the following more general problem:

Problem 14

Given recurrence relation

$$a_n = Aa_{n-1} + Ba_{n-2} \qquad \textit{with } a_0, a_1 \textit{ known}$$

with known coefficients A and B, determine a closed formula for a_n for all $n > 1$.

fib_recursive .py

```
def fib(n):
    if n==0 or n==1: return n
    return fib(n-1) + fib(n-2)

for k in [0,1,6,10,30]:
    print("fib(%s) = %s" % (k,fib(k)))
```

```
fib(0) = 0
fib(1) = 1
fib(6) = 8
fib(10) = 55
fib(30) = 832040
```

The Characteristic Root Technique

STEP 1 Given recurrence relation

$$a_n = Aa_{n-1} + Ba_{n-2}$$

we **assume** that the solution will look like $a_n = \lambda^n$ for some, as yet, unknown λ.

STEP 2 We substitute our guess into the recurrence relation and simplify to get an equation involving a polynomial (a quadratic) — called the characteristic equation.

STEP 3 We solve the characteristic equation for λ and build solution using the following rules:

- If the characteristic polynomial has two distinct solutions, say $\lambda = \lambda_1$ and $\lambda = \lambda_2$, then the solution to the recurrence relation looks like

$$a_n = C\lambda_1^n + D\lambda_2^n,$$

- If the characteristic equation has one repeated solutions, say $\lambda = \lambda_1 = \lambda_2$, then the solution to the recurrence relation looks like

$$a_n = C\lambda_1^n + Dn\lambda_1^n$$

STEP 4 Use the known values of a_0 and a_1 to determine the coefficients C and D.

Example — Distinct Solutions to Characteristic Equation I

Example 15

Solve the recurrence relation $a_n = 7a_{n-1} - 10a_{n-2}$ with $a_0 = 2$ and $a_1 = 3$.

Solution. First write recurrence equation as

$$a_n - 7a_{n-1} + 10a_{n-2} = 0$$

STEP 1 Try solution of the form $a_n = \lambda^n$

$$a_n = \lambda^n \qquad \Longrightarrow \qquad a_{n-1} = \lambda^{n-1} \quad \text{and} \quad a_{n-2} = \lambda^{n-2}$$

STEP 2 Substituting a_n, a_{n-1} and a_{n-2} into the recurrence equation ...

$$\begin{aligned} a_n - 7a_{n-1} + 10a_{n-2} = 0 &\Longrightarrow \left[\lambda^n\right] - 7\left[\lambda^{n-1}\right] + 10\left[\lambda^{n-2}\right] = 0 \\ &\Longrightarrow \left[\lambda^2\lambda^{n-2}\right] - 7\left[\lambda\lambda^{n-2}\right] + 10\left[\lambda^{n-2}\right] = 0 \\ &\Longrightarrow \lambda^{n-2}\left[\lambda^2 - 7\lambda + 10\right] = 0 \\ &\Longrightarrow \lambda^2 - 7\lambda + 10 = 0 \end{aligned}$$

characteristic equation

Example — Distinct Solutions to Characteristic Equation II

STEP 3 Solve the characteristic equation … and build solution

$$\lambda^2 - 7x\lambda + 10 = 0 \implies (\lambda - 2)(\lambda - 5) = 0$$

We have two distinct roots, $\lambda = 2$ and $\lambda = 5$ so we know the solution looks like

$$a_n = C2^n + D5^n$$

where C and D will be determined using the initial conditions.

STEP 4 Determine coefficients C and D using $a_0 = 2$ and $a_1 = 3$ …

Plug in $n = 0$ and $n = 1$ into a_n to get a system of two equations with two unknowns, which we solve

$$\left.\begin{array}{l} 2 = a_0 = C2^0 + D5^0 = C + D \\ 3 = a_1 = C2^1 + D5^1 = 2C + 5D \end{array}\right\} \implies C = \frac{7}{3} \quad \text{and} \quad D = -\frac{1}{3}$$

So the solution to the recurrence relation is

$$a_n = \frac{7}{3}2^n - \frac{1}{3}3^n.$$

Example — Repeated Solution to Characteristic Equation I

Example 16

Solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 4$.

Solution. First write recurrence equation as

$$a_n - 6a_{n-1} + 9a_{n-2} = 0$$

STEP 1 Try solution of the form $a_n = \lambda^n$

$$a_n = \lambda^n \quad \Longrightarrow \quad a_{n-1} = \lambda^{n-1} \quad \text{and} \quad a_{n-2} = \lambda^{n-2}$$

STEP 2 Substituting a_n, a_{n-1} and a_{n-2} into the recurrence equation ...

$$\begin{aligned} a_n - 6a_{n-1} + 9a_{n-2} = 0 &\Longrightarrow \left[\lambda^n\right] - 6\left[\lambda^{n-1}\right] + 9\left[\lambda^{n-2}\right] = 0 \\ &\Longrightarrow \left[\lambda^2\lambda^{n-2}\right] - 6\left[\lambda\lambda^{n-2}\right] + 9\left[\lambda^{n-2}\right] = 0 \\ &\Longrightarrow \lambda^{n-2}\left[\lambda^2 - 6\lambda + 9\right] = 0 \\ &\Longrightarrow \lambda^2 - 6\lambda + 9 = 0 \end{aligned}$$

characteristic equation

Example — Distinct Solutions to Characteristic Equation II

STEP 3 Solve the characteristic equation ... and build solution

$$\lambda^2 - 6\lambda + 9 = 0 \implies (\lambda - 3)^2 = 0$$

We have a repeated root, $\lambda = 3$ so we know the solution looks like

$$a_n = C3^n + Dn3^n$$

where C and D will be determined using the initial conditions.

STEP 4 Determine coefficients C and D using $a_0 = 1$ and $a_1 = 4$ ...

Plug in $n = 0$ and $n = 1$ into a_n to get a system of two equations with two unknowns, which we solve

$$\left.\begin{array}{l} 1 = a_0 = C3^0 + D(0)(3^0) = a = C \\ 4 = a_1 = C(3^1) + D(1)(3^1) = 3C + 3D \end{array}\right\} \implies C = 1 \quad \text{and} \quad D = \frac{1}{3}$$

So the solution to the recurrence relation is

$$a_n = 3^n + \frac{1}{3}n3^n.$$

Review Exercises 3 (Solving Recurrence Relations)

Question 1:
Show that 4^n is a solution to the recurrence relation $a_n = 3a_{n-1} + 4a_{n-2}$.

Question 2:
Determime the solution to the recurrence relation $a_n = 3a_{n-1} + 4a_{n-2}$ with initial terms $a_0 = 2$ and $a_1 = 3$

Question 3:
Determime the solution to the recurrence relation $a_n = 3a_{n-1} + 4a_{n-2}$ with initial terms $a_0 = 5$ and $a_1 = 8$.

Question 4:
Solve the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$.

(a) What is the solution if the initial terms are $a_0 = 1$ and $a_1 = 2$?

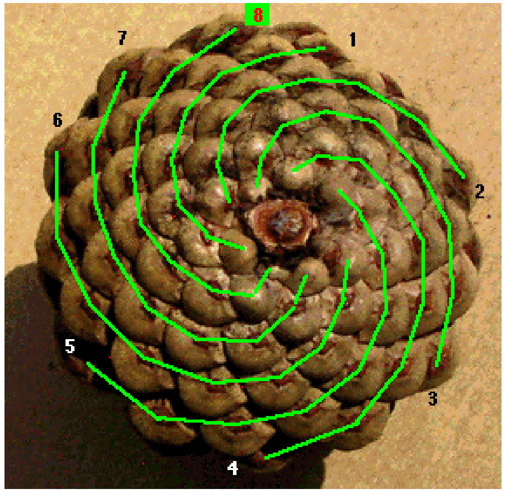
(b) What do the initial terms need to be in order for $a_9 = 30$?

(c) For which x are there initial terms which make $a_9 = x$?

Question 5:
Determine the closed formula for the Fibonacci sequence.

Fibonacci Sequence in Nature and man made

Pine cone



Fibonacci Spirals